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Connes' noncommutative differential geometry and the Standard Model

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In this paper, the Connes–Lott approach to the phenomenological Lagrangian of the standard theory of elementary particles is reviewed in detail. The paper is self-contained, in that the necessary foundations in noncommutative geometry are fully laid out.

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To the memory of Robert Marshak

Contents

Introduction

The purpose of this paper is to give an account of Connes' noncommutative geometry, insofar as it is used to "derive" the classical field theory Lagrangian of the Standard Model of particle physics. Previously published [10–12] or widely circulating [29] treatments are sketchy, either in the physics or in the mathematics or in both, so we deem that a more detailed and updated treatment is warranted.

We review the mathematics of noncommutative geometry in sections 1 to 5. Prerequisites on bundles of spinors, Dirac operators and the like are confined to the appendix. The final three sections deal with the application to the Standard Model. This paper does not give a treatment of all branches of noncommutative geometry; in particular, cyclic cohomology is not emphasized.

Section 1 introduces the Dixmier trace, which has become an essential mathematical tool of noncommutative geometry. Section 2 gives a new proof of Connes' trace theorem, which is the bridge between commutative and noncommutative integration theories. In sections 3 and 4, we give the algebraic underpinnings of noncommutative geometry: universal forms and connections and the relevant cohomology theories are introduced.

Section 5 is the heart of the theory; we introduce K-cycles and the π_D homomorphism, and we identify the algebra of differential forms on a spin^c manifold with an algebra of classes of operators on the Hilbert space of the Dirac K-cycle.

In section 6, we introduce the Yang–Mills and fermionic actions in noncommutative geometry. We exhibit the Hochschild cocycle, which, in the commutative case, gives the usual lower bound for the Yang–Mills action. Gauge invariance is examined in the noncommutative context. We also touch on the matter of Poincaré duality.

The Glashow–Weinberg–Salam model for electroweak interaction of leptons is derived in section 7 from the product K-cycle of the Dirac K-cycle on a compactified spacetime and a K-cycle for a two-point space, which is a carrier for the mass of elementary particles.

In section 8, the full Standard Model is developed. Colour symmetry appears by introducing a supplementary bimodule structure on the K-cycle. We touch on the question of whether the Connes–Lott approach gives new information on the parameters of the Standard Model. The hypercharge assignments for the several fermions are derived.

From the philosophical point of view, it can be argued that what we do is to interpret geometrically the intricacies of an accurate phenomenological model: high-energy physics would in fact be the unveiling of the fine structure of spacetime. In particular, the Higgs boson would be another gauge field, corresponding to a coupling among the leaves of spacetime. In this way, the fact that the Higgs fields behave rather like Yang–Mills fields in that they are self-interacting and have a "pointlike" interaction with fermions receives a surprising geometrical explanation.

Some outstanding problems remain. The choice of K-cycles is somewhat empirical. K-cycles on spaces with indefinite metric have not been studied. Lastly, no one seems to know how to quantize the action within the framework of noncommutative geometry.

1. Ideals of operators and Dixmier traces

1.1. We recall some facts about compact operators on Hilbert spaces that we will need. Suppose $\mathcal{H}$ is a separable infinite-dimensional Hilbert space, and the operator $A \in \mathcal{L}(\mathcal{H})$, the space of bounded linear operators on $\mathcal{H}$, is compact. Then A has a uniformly convergent expansion:

$$A = \sum_{j\geq 0} s_j(A)\, |\psi_j\rangle\langle\phi_j|, \tag{1.1}$$

where each $s_j > 0$, with $s_0(A) \geq s_1(A) \geq \cdots$, and $\{\psi_j\}$, $\{\phi_j\}$ are orthonormal sets. This is easily proven using polar decomposition $A = U|A|$, where $|A| := (A^\dagger A)^{1/2}$ is compact self-adjoint; the $s_j(A)$ are the nonzero eigenvalues of $|A|$ (with repeated multiplicity), the ϕ_j are its eigenvectors and the $\psi_j = U\phi_j$ are the eigenvectors of $(AA^\dagger)^{1/2}$.

The $s_j(A)$, called the *singular values* of A, play the main role in the theory of ideals that follows. Also of importance are their partial sums:

$$\sigma_n(A) := \sum_{j=0}^{n-1} s_j(A). \tag{1.2}$$

One needs several inequalities satisfied by the $s_j(A)$ and the $\sigma_n(A)$. Many of them are established from the formulae:

$$s_j(A) = \inf\{\, \|A - T\| : T \in \mathcal{K}_j \,\}, \tag{1.3a}$$

$$\sigma_n(A) = \sup\{\, \|AE\|_1 : \dim\operatorname{im}(E) = n \,\}. \tag{1.3b}$$

Here $\mathcal{K}_j$ denotes the set of operators of rank at most j, E an orthogonal projector, and $\|\cdot\|_1$ denotes the trace norm.

For instance, from (1.3a) and the trivial $\mathcal{K}_j + \mathcal{K}_k \subset \mathcal{K}_{j+k}$, one gets

$$s_{j+k}(A+B) \leq s_j(A) + s_k(B), \qquad s_{j+k}(AB) \leq s_j(A)s_k(B). \tag{1.4}$$

[For the second inequality, apply (1.3a) after noting that if $T_1 \in \mathcal{K}_j$, $T_2 \in \mathcal{K}_k$, then $(A - T_1)(B - T_2) = AB - T_3$ with $T_3 = T_1B + (A - T_1)T_2 \in \mathcal{K}_{j+k}$.] From (1.3b) one has immediately

$$\sigma_n(A+B) \leq \sigma_n(A) + \sigma_n(B), \tag{1.5}$$

i.e., the σ_n are norms.

Another useful result is *Weyl's theorem*:

$$\sum_{j=0}^{n-1} |\lambda_j(A)| \leq \sigma_n(A), \tag{1.6}$$

where $\lambda_j(A)$ are the *eigenvalues* of A, arranged in order of decreasing absolute value and counted with (algebraic) multiplicity. This needs some exterior algebra for a quick proof [37, p. 11].

1.2. Suppose $\mathcal{L}$ is a nontrivial ideal in $\mathcal{L}(\mathcal{H})$. A norm $|||\cdot|||$ on $\mathcal{L}$ is called *symmetric* if

$$|||ABC||| \leq \|A\|\,|||B|||\,\|C\| \quad \text{for } B \in \mathcal{L},\ A, C \in \mathcal{L}(\mathcal{H}). \tag{1.7}$$

We say that $\mathcal{L}$ is a *symmetrically normed ideal* if $\mathcal{L}$ is complete in the norm $|||\cdot|||$. The theory of symmetrically normed ideals may be found in Gohberg and Kreĭn [21], whose treatment we summarize here.

Examples are the well-known Schatten ideals $\mathcal{L}^p(\mathcal{H})$ for $1 \leq p < \infty$ and the maximal ideal $\mathcal{L}^\infty(\mathcal{H})$ of all compact operators. If $\mathcal{K}$ denotes the set of finite-rank operators, then $\mathcal{K} \subset \mathcal{L} \subseteq \mathcal{L}^\infty$. Some trickier examples, in particular the Dixmier trace class, are of paramount importance in noncommutative geometry.

It is not hard to show—using (1.7) and polar decomposition—that the symmetric norms of A and of $|A|$ are equal, and hence that $|||A|||$ depends only on the singular values of A. Thus we have, for finite-rank operators:

$$|||A||| = \Phi(s_0(A), s_1(A), s_2(A), \dots), \tag{1.8}$$

where Φ is a function, with nonnegative real values, defined on the cone $\mathbf{k}_{00}$ of sequences of nonnegative numbers $(x_0, \dots, x_n, 0, 0, \dots)$ satisfying $x_j \geq x_{j+1}$ for all j which are zero after finitely many terms. The necessary and sufficient conditions on such a function Φ to yield a symmetric norm are:

(i) $\Phi(\alpha x) = \alpha\Phi(x)$ if $x \in \mathbf{k}_{00}$, $\alpha > 0$;
(ii) $\Phi(x + y) \leq \Phi(x) + \Phi(y)$ if $x, y \in \mathbf{k}_{00}$;
(iii) $\Phi(x) > 0$ if $x \in \mathbf{k}_{00}$, $x \neq 0$;
(iv) $\Phi(1, 0, 0, \dots) = 1$;
(v) if $\sum_{k=0}^{n-1} x_k \leq \sum_{k=0}^{n-1} y_k$ for all n, then $\Phi(x) \leq \Phi(y)$.

A function on $\mathbf{k}_{00}$ satisfying (i)–(v) is called a *gauge*. Property (v) holds since, if A, B are compact positive operators with x, y as respective sets of eigenvalues, then $A = CB$ where C is an operator with $\|C\| \leq 1$, so that $|||A||| \leq |||B|||$.

Let us denote by $\mathbf{k}_0$ the cone of nonincreasing sequences of nonnegative numbers which tend to zero. We then define

$$\mathbf{k}_\Phi := \{\, x \in \mathbf{k}_0 : \Phi(x) := \sup_n \Phi(x_1, \dots, x_n, 0, 0, \dots) < \infty \,\},$$
$$\mathcal{L}^\Phi := \{\, A \in \mathcal{L}^\infty(\mathcal{H}) : (s_0(A), s_1(A), s_2(A), \dots) \in \mathbf{k}_\Phi \,\}. \tag{1.9}$$

Then $\mathcal{L}^\Phi$ is a symmetrically normed ideal; its norm given by (1.8), whose right hand side is of the form $\Phi(x)$ with $x \in \mathbf{k}_\Phi$. That $\mathcal{L}^\Phi$ is a vector space is due to the inequality (1.5).

Notice that the gauge $\Phi_\infty(x) := x_1$ gives $\mathcal{L}^\Phi = \mathcal{L}^\infty(\mathcal{H})$, in which case (1.8) is the usual operator norm; $\Phi_1(x) := \sum_{j=0}^\infty x_j$ gives the trace-class operators $\mathcal{L}^1(\mathcal{H})$, and the Schatten classes $\mathcal{L}^p(\mathcal{H})$ come (by definition) from $\Phi_p(x) := (\sum_j x_j^p)^{1/p}$. The corresponding symmetric norms are traditionally written $\|\cdot\|_p$. The class $\mathcal{L}^2(\mathcal{H})$ is the Hilbert–Schmidt class. All of these ideals are *separable*, which is to say, the finite-rank operators $\mathcal{K}$ are dense in each.

From the properties of a gauge, it is easily seen that $\Phi_\infty(x) \le \Phi(x) \le \Phi_1(x)$, so that Φ_∞ is minimal and Φ_1 is maximal among all gauges. In consequence, we have that $\mathcal{L}^1 \subseteq \mathcal{L} \subseteq \mathcal{L}^\infty$ for any symmetrically normed ideal $\mathcal{L}$.

There exist symmetrically normed ideals of compact operators which are not separable, i.e., in which the finite-rank operators are not dense. Let A_m denote the finite-rank operator $\sum_{j=0}^m s_j(A)\,|\psi_j\rangle\langle\phi_j|$; clearly $A_m \in \mathcal{L}^\Phi$ whenever $A \in \mathcal{L}^\Phi$. The finite-rank operators are dense iff $|||A - A_m||| \to 0$ as $m \to \infty$. Thus $\mathcal{L}^\Phi$ is not separable iff

$$\lim_{m\to\infty} \Phi(x_{m+1}, x_{m+2}, \dots) \neq 0, \tag{1.10}$$

whenever $x \in \mathbf{k}_\Phi$. We shall give explicit examples soon. We denote by $\mathcal{L}_0^\Phi$ the closure of the finite-rank operators, which is a proper closed subspace if $\mathcal{L}^\Phi$ is not separable.

Two symmetrically normed ideals coincide: $\mathcal{L}^\Phi = \mathcal{L}^\Psi$ algebraically (and also topologically, by the closed graph theorem) iff Φ and Ψ are equivalent in the sense that:

$$C_1\Psi(x) \le \Phi(x) \le C_2\Psi(x), \quad \text{with } 0 < C_1 \le C_2 < \infty. \tag{1.11}$$

The *dual space* of a symmetrically normed ideal can be identified as follows. We define the "dual gauge" of Φ to be the function:

$$\Phi'(y) := \sup\left\{ \frac{\langle x, y\rangle}{\Phi(x)} : x \in \mathbf{k}_{00},\ x \neq 0 \right\}, \tag{1.12}$$

where $\langle x, y\rangle := \sum_k x_k y_k$. One checks that Φ' is also a gauge; property (iv) comes from the inequality $x_1 \le \Phi(x)$. Since $\langle x, y\rangle = \sup\{\,\Phi(x)\Phi'(y) : x \in \mathcal{L}^\Phi,\ y \in \mathcal{L}^{\Phi'}\}$, we see that $\Phi'' = \Phi$. Then one can show that, unless Φ is equivalent to Φ_1, the dual of $\mathcal{L}_0^\Phi$ is isometric to $\mathcal{L}^{\Phi'}$, and moreover the duality is given by functionals of the form $A \mapsto \operatorname{Tr}[AB]$, for $B \in \mathcal{L}^{\Phi'}$. The well-known duality of the Schatten ideals $\mathcal{L}^p$ and $\mathcal{L}^{p'}$ when $1/p + 1/p' = 1$ is a special case of this.

1.3. We can now produce examples of nonseparable ideals. Suppose Π is a sequence of positive numbers with $\pi_j \ge \pi_{j+1}$ for all j and $\pi_1 = 1$. Define:

$$\Phi_\Pi(x) := \sup_n \left\{ \sum_{j=0}^{n-1} x_j \Big/ \sum_{j=0}^{n-1} \pi_j \right\} \tag{1.13}$$

for $x \in \mathbf{k}_{00}$. Then Φ_Π satisfies (i)–(v) and so defines an ideal $\mathcal{L}_\Pi := \mathcal{L}^{\Phi_\Pi}$. In general it turns out that there are three cases:

(i) $\sum_j \pi_j < \infty$: then Φ_Π is equivalent to Φ_1, and $\mathcal{L}_\Pi = \mathcal{L}^1$;

(ii) $\lim_j \pi_j > 0$: then Φ_Π is equivalent to Φ_∞, and $\mathcal{L}_\Pi = \mathcal{L}^\infty$;

(iii) $\sum_j \pi_j = \infty$ and $\lim_j \pi_j = 0$: then (1.10) holds, since $\Phi(\pi_{m+1}, \pi_{m+2}, \ldots) \to 1$, and so $\mathcal{L}_\Pi$ is not separable.

We can moreover identify the dual space for case (iii) ideals. Let Φ'_Π denote the function

$$\Phi'_\Pi(x) := \langle \Pi, x \rangle = \sum_{j=0}^{\infty} \pi_j x_j. \tag{1.14}$$

From (1.12), the dual gauge to Φ'_Π is just Φ_Π (hence the notation); also, since the tail of a convergent positive series vanishes, the associated ideal $'\mathcal{L}_\Pi$ is separable. From the remarks after (1.12), we have that $'\mathcal{L}_\Pi$ is the dual space of $\mathcal{L}_{\Pi,0}$, and $\mathcal{L}_\Pi$ is the dual space of $'\mathcal{L}_\Pi$. Since $\mathcal{L}_\Pi$ is not separable, but $\mathcal{L}_{\Pi,0}$ is, these three spaces are distinct, and they are not reflexive.

Finally, take $\pi_j := 1/(j+1)$, the harmonic sequence, obviously belonging to case (iii). As the harmonic series is asymptotically $\log n + \gamma$, with γ the Euler–Mascheroni constant, we can replace the denominator in (1.13) by $\log n$.

Lemma 1.1. *Let $1 < p < \infty$ and let $p' := p/(p-1)$. Denote the norms in $\mathcal{L}_\Pi$ and $'\mathcal{L}_\Pi$ by $|||\cdot|||_\Pi$ and $|||\cdot|||_{\Pi'}$, respectively. Then*

$$|||A|||_{\Pi'} \le \|\Pi\|_{p'} \|A\|_p, \quad \text{and} \quad \|A\|_p \le \|\Pi\|_p |||A|||_\Pi, \tag{1.15}$$

where $\|\Pi\|_p = \zeta(p)^{1/p}$ is the usual sequence-space norm. Therefore:

$$\mathcal{L}^1 \subset \mathcal{L}_\Pi \subset \mathcal{L}^p \subset {'\mathcal{L}_\Pi} \subset \mathcal{L}^\infty, \quad \text{for all } 1 < p < \infty, \tag{1.16}$$

where the inclusions are continuous, and all but the first are dense.

Proof. Hölder's inequality gives $|||A|||_{\Pi'} = \sum_j \pi_j s_j(A) \le \|\Pi\|_{p'} \|A\|_p$, yielding the first estimate for $A \in \mathcal{L}^p$.

On the other hand, if $A \in \mathcal{L}_\Pi$, we have

$$\sum_{j=0}^{n-1} s_j(A) \le |||A|||_\Pi \sum_{j=0}^{n-1} \pi_j$$

for all n. Since the function $t \mapsto t^p$ is convex, we can apply it to both sides to obtain

$$\sum_{j=0}^{n-1} s_j(A)^p \le |||A|||_\Pi^p \sum_{j=0}^{n-1} \pi_j^p,$$

and so $\|A_n\|_p \le |||A|||_\Pi \|\Pi\|_p$. This uniform bound on $\|A_n\|_p$ shows that $A \in \mathcal{L}^p$ with $\|A_n - A\|_p \to 0$. Letting $n \to \infty$ gives the second estimate of (1.15). □

1.4. We consider now a special two-parameter family of ideals introduced by Connes [13], denoted $\mathcal{L}^{p,q}(\mathcal{H})$ or simply $\mathcal{L}^{p,q}$, where $1 < p < \infty$ and $1 \leq q \leq \infty$. For $q < \infty$, a compact operator A belongs to $\mathcal{L}^{p,q}$ iff

$$\sum_{n=1}^{\infty} \frac{\sigma_n(A)^q}{n^{1+q/p'}} < \infty. \tag{1.17}$$

For $q = \infty$: a compact operator A belongs to $\mathcal{L}^{p,\infty}$ iff the sequence $n^{-1/p'}\sigma_n(A)$ is bounded—which is equivalent to $s_n(A) = \mathrm{O}(n^{-1/p})$.

A corresponding gauge is

$$\Phi(x) := \zeta(1 + q/p')^{-1/q} \left(\sum_n n^{-(1+q/p')}(x_1 + \cdots + x_n)^q \right)^{1/q};$$

it clearly satisfies all the requirements for a gauge except perhaps for (ii), which follows from Minkowski's inequality. Thus the class of operators satisfying (1.17) is indeed a symmetrically normed ideal $\mathcal{L}^{\Phi}$.

It is possible to show that $\mathcal{L}^{p_1,q_1} \subset \mathcal{L}^{p_2,q_2}$ with strict inclusion if $p_1 < p_2$ or $p_1 = p_2$ and $q_1 < q_2$ and that $\mathcal{L}^{p,p}$ is the standard Schatten class $\mathcal{L}^p$. The $\mathcal{L}^{p,\infty} =: \mathcal{L}^{p+}$ are called weak-$\mathcal{L}^p$ spaces by Simon [37].

Defining $\alpha := 1/p$ and $\beta := 1/q$, one can nicely represent those ideals as points (α, β) of a square of side equal to 1, with the Schatten classes on the diagonal. Duality is symmetry with respect to the middle point of the square, except on the boundary, where we indeed have that $\mathcal{L}^{p+}$ is the dual of $\mathcal{L}^{p',1}$, but the latter is not the dual of the *nonseparable* $\mathcal{L}^{p+}$, but of the norm closure $\mathcal{L}_0^{p+}$ of $\mathcal{K}$, with respect to the norm

$$\|A\|_{p+} := \sup_n \frac{\sigma_n(A)}{n^{(p-1)/p}}. \tag{1.18}$$

One has $A \in \mathcal{L}_0^{p+}$ iff $s_n(A) = \mathrm{o}(n^{-1/p})$.

Notice that the vertical sides of the square are void. We are nevertheless going to identify the four corners: $\mathcal{L}^{\infty,\infty} := \mathcal{L}^{\infty}$ and $\mathcal{L}^{1,1} := \mathcal{L}^1$, to begin with. Also, $\mathcal{L}^{1+}$ is the Dixmier trace class ideal—or Dixmier ideal, for short—of operators A such that:

$$\|A\|_{1+} := \sup_n \frac{\sigma_n(A)}{\log n} < \infty. \tag{1.19}$$

This is just $\mathcal{L}_{\Pi}$ where Π is the harmonic sequence, so it is the dual of the so-called Macaev ideal [32]: $\mathcal{L}^{\infty,1} = {}'\mathcal{L}_{\Pi}$. For any $A \in \mathcal{L}^{p+}$, $B \in \mathcal{L}^{p'+}$, we have $AB \in \mathcal{L}^{1+}$.

Notice that $\mathcal{L}_0^{1+}$ is the set of compact operators such that the order of growth of the partial sums of singular values is less than logarithmic. In the chink just above it, Dixmier traces will lodge themselves.

1.5. We come, then, to the question of the trace. If A is a positive operator in the Dixmier ideal $\mathcal{L}^{1+}$, we would like to define a positive functional Tr^+ by

$$\mathrm{Tr}^+ A = \lim_{n\to\infty} \frac{\sigma_n(A)}{\log n}, \tag{1.20}$$

on positive operators with the trace property, and extend it also to nonpositive operators in $\mathcal{L}^{1+}$. Note that Weyl's inequality (1.6) guarantees that $(\log n)^{-1} \times \sum_{j=1}^{n} \lambda_j(A)$ is a bounded sequence if $\sigma_n(A)/\log n$ is bounded.

An evident but crucial property of the trace Tr^+ will be that it vanishes on the ordinary trace-class operators $\mathcal{L}^1$, since $\sum_j s_j(A)$ is convergent. (This is why the nondensity of the first inclusion in (1.16) is important.)

There are two difficulties with formula (1.20). Obviously the "limit" involved must be some sort of generalized limit process which should be meaningful for all bounded sequences and not just for convergent sequences. Moreover, it should be such that the right hand side of (1.20) is an additive functional on the positive cone of $\mathcal{L}^{1+}$.

Write $\gamma_n(A) := \sigma_n(A)/\log n$ for $A \in \mathcal{L}^{1+}$ positive. From (1.5) we get $\gamma_n(A+B) \le \gamma_n(A) + \gamma_n(B)$. Now (1.3b) simplifies to $\sigma_n(A) = \sup\{\mathrm{Tr}(AE) : \dim \operatorname{im}(E) = n\}$ for positive A, thus $\sigma_n(A) + \sigma_n(B) \le \sigma_{2n}(A+B)$, and so $\gamma_n(A) + \gamma_n(B) \le \gamma_{2n}(A+B)(1 + \log 2/\log n)$. Thus (1.20) will be additive provided the generalized limit involved satisfies the scaling property $\lim \gamma_{2n}(A) = \lim \gamma_n(A)$.

Dixmier [16] noticed that generalized limits with the right type of scale invariance are indeed available. One replaces the formal limit $\lim_{n\to\infty}$ by a *mean* $\lim_\omega$ on the space $\ell^\infty(\mathbb{N})$ of bounded sequences, which is a linear and positive (hence continuous) functional equalling the usual limit for convergent sequences, and satisfying $\lim_\omega \tilde{x}_n = \lim_\omega x_n$ for $x \in \ell^\infty(\mathbb{N})$, where $\tilde{x} = (x_0, x_0, x_2, x_2, \dots, x_{2n}, x_{2n}, \dots)$. Define $f_x \in L^\infty(\mathbb{R})$ by $f_x(t) := x_n$ whenever $n \le t < n+1$ or $-n-1 \le t < -n$. The affine group $M(1)$ acts on $\mathbb{R}$ and hence on $L^\infty(\mathbb{R})$ by translations and dilations $t \mapsto at + b$ ($a \ne 0$, $b \in \mathbb{R}$). Now $M(1)$ is *amenable*, that is to say, there is a positive linear (thus continuous) functional ω on $L^\infty(\mathbb{R})$ with $\omega(1) = 1$ which is $M(1)$-invariant; indeed, there are an infinity of such invariant means. Let $\lim_\omega x_n := \omega(f_x)$.

Clearly $\lim_\omega$ is a positive linear functional on $\ell^\infty(\mathbb{N})$. From the translation invariance of ω, we get $\lim_\omega x_n = 0$ if f_x vanishes at infinity, i.e., if $x_n \to 0$. Also, $\lim_\omega$ of the constant sequence 1 is $\omega(1) = 1$; thus $\lim_\omega x_n$ is the ordinary limit when x is a convergent sequence. Finally, the invariance of ω under $t \mapsto 2t$ gives $\lim_\omega x_{2n} = \lim_\omega \tilde{x}_n$. Since

$$\gamma_n(A) - \gamma_{n+1}(A) = \left(\frac{\log(n+1)}{\log n} - 1\right)\gamma_{n+1}(A) - \frac{s_{n+1}(A)}{\log n}, \tag{1.21}$$

we have $\gamma_n(A) - \gamma_{n+1}(A) \to 0$ as $n \to \infty$. With $x_n := \gamma_n(A)$, we then have that $\tilde{x}_n - x_n \to 0$ as $n \to \infty$. Therefore $\lim_\omega x_n = \lim_\omega \tilde{x}_n = \lim_\omega x_{2n}$, i.e.,

$\lim_\omega \gamma_n(A) = \lim_\omega \gamma_{2n}(A)$. Also, $\lim_\omega \gamma_n(A)$ has the property of unitary invariance (since it depends only on the eigenvalues of the positive operator A), so it extends by linearity to a *trace* Tr_ω on the ideal $\mathcal{L}^{1+}$.

The linear functionals $\lim_\omega$ on $\ell^\infty(\mathbb{N})$ lack the property of countable additivity, like the means ω themselves, and thus the traces Tr_ω are not ultraweakly continuous, and so cannot be explicitly computed in general. This is not a major obstacle, however: whenever one can show that $\gamma_n(A) = y_n + z_n$, where y is a (computable) *convergent* sequence, and $z_n = \gamma_n(B)$ for some $B \in \mathcal{L}_0^{1+}$, then $\mathrm{Tr}_\omega(A)$ is the *ordinary* limit $\lim_{n\to\infty} y_n$. This turns out to be the case of interest for noncommutative geometry. We will therefore continue to use the notation Tr^+ and the expression (1.20) for suitable $A \in \mathcal{L}^{1+}$.

2. The trace theorem

2.1. The Dirac operator on a spin^c manifold M is a particular example of an elliptic pseudodifferential operator. (We refer to the appendix for the definition and properties of the Dirac operator. We shall here and henceforth suppose that M is a compact smooth manifold, without boundary.)

To any pseudodifferential operator on a compact manifold, we can associate a matrix-valued function on the cotangent bundle, called its *principal symbol.* If the operator is of order $-\dim M$, one can define a certain integral of the matrix trace of this function, called the *residue* of the operator. At the heart of noncommutative geometry lies the trace theorem of Connes, which says that the Dixmier trace of such an operator equals its residue.

2.2. We first review briefly the standard facts about pseudodifferential operators on manifolds [31, 39]. A differential operator on a compact manifold M is a linear map $P\colon \Gamma(E) \to \Gamma(E)$ on sections of a vector bundle E over M—of rank k, say—which can be written, in local coordinates for suitable trivializations of E, as $P = \sum_{|\alpha|\le m} f_\alpha(x)(-i)^{|\alpha|}\partial^{|\alpha|}/\partial x^\alpha$, where each f_α is a $k\times k$ matrix of smooth functions on M and m is the order of P.

The "complete symbol" of P is

$$\sum_{j=0}^{m} p_{m-j}(x,\xi), \quad p_{m-j} = \sum_{|\alpha|=m-j} f_\alpha(x)\xi^\alpha.$$

The leading term in this sum is the *principal symbol*

$$p_m(x,\xi) := \sum_{|\alpha|=m} f_\alpha(x)\xi^\alpha.$$

The operator P is then given on local sections by

$$\begin{aligned} Pu(x) &:= (2\pi)^{-n/2} \int e^{i\langle\xi,x\rangle} p(x,\xi)\hat{u}(\xi)\,d\xi \\ &= (2\pi)^{-n} \int e^{i\langle\xi,x-y\rangle} p(x,\xi)u(y)\,d\xi\,dy. \end{aligned} \tag{2.1}$$

The principal symbol has an invariant meaning. Let $\pi\colon T^*M \to M$ be the canonical projection for the cotangent bundle. We can use it to pull back E to the bundle π^*E, with the same fibres as before, over T^*M. The principal symbol determines a bundle homomorphism of π^*E, i.e., a section p_m of the bundle $\pi^*\operatorname{End}(E)$ over T^*M, which is clearly the pullback of $\operatorname{End}(E)$ by the cotangent projection:

$$\begin{array}{ccc} \pi^*\operatorname{End}(E) & \longrightarrow & \operatorname{End}(E) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \sigma} \\ T^*M & \xrightarrow{\pi} & M \end{array} . \tag{2.2}$$

A so-called classical pseudodifferential operator P of order m (on a trivial vector bundle over R^n) has also a "complete symbol" $p(x,\xi)$ which has an asymptotic expansion of the form

$$p(x,\xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x,\xi), \tag{2.3}$$

where now m can be any complex number, and the p_{m-j} are matrices of smooth functions, homogeneous in ξ of degree $(m-j)$, i.e.,

$$p_{m-j}(x,\lambda\xi) = \lambda^{m-j} p_{m-j}(x,\xi).$$

Some additional conditions are usually imposed to control the growth of p in the x variables. We shall consider only classical pseudodifferential operators in the sequel. We will write $\sigma_m(P) := p_m$ to denote the principal symbol of P.

An important formula [39] relates the complete symbol of an operator product $R = PQ$ to those of the factors by the expansion

$$r(x,\xi) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{i^{-|\alpha|}}{\alpha!} (\partial_\xi^\alpha p)(\partial_x^\alpha q). \tag{2.4}$$

In particular, the leading term $|\alpha| = 0$ of (2.4) shows that the principal symbol of an operator product is the product of the principal symbols of the factors:

$$r_{m+m'}(x,\xi) = p_m(x,\xi)\, q_{m'}(x,\xi), \tag{2.5}$$

where the order of PQ is $m + m'$, since the right hand side is homogeneous of this degree in ξ. Furthermore, the complete symbol of the adjoint operator $P^\dagger$ is a complicated expression, but its principal symbol is just the hermitian conjugate $p_m^*(x,\xi)$. As a consequence of all this, the principal symbol of a positive pseudodifferential operator $R = P^\dagger P$ is nonnegative.

Pseudodifferential operators of sufficiently low degree are integral operators of the form: $Pu(x) = \int k(x,y)u(y)\,d\mu(y)$, where k is locally summable. When k is smooth, P is a "smoothing operator" of order $-\infty$; two pseudodifferential operators with the same asymptotic expansion can differ by a smoothing operator.

Another regularity property of pseudodifferential operators is that they satisfy L^2 estimates of various kinds. We mention only two of these, which we will need: an operator of order 0 is bounded on the Hilbert space $L^2(E)$ of square-integrable sections, and an operator of order $-n$ is compact. For the proofs, we refer to refs. [31, 39].

2.3. An operator P acting on sections of a vector bundle E on a manifold M is a pseudodifferential operator of order m, by definition, iff $s \mapsto \phi P(\psi s)$ is a pseudodifferential operator of order m whenever $\phi, \psi \in C^\infty(M)$ are supported in trivializing charts for E. Any P may be reconstructed from such components by suitable partitions of unity.

A pseudodifferential operator is called *elliptic* if its principal symbol is a bundle isomorphism off the zero section of T^*M. Supposing M to be a Riemannian manifold, so that $\|\xi\|^2$ is defined, this means that the linear transformation $p_m(x,\xi)$ of the fibres of E is invertible on the "cosphere bundle" $\mathbb{S}^*M := \{ (x,\xi) \in T^*M : \|\xi\| = 1 \}$, since p_m is homogeneous in ξ.

By proposition A.7 of the appendix, the principal symbol of the Dirac operator D is the operator $c(\xi)$ of left Clifford multiplication by ξ, and so D is a first-order elliptic operator.

The *Laplacian* $\Delta = -D^2$ has principal symbol $-\|\xi\|^2$, and so is a second-order elliptic operator. The resolvent operator $(-\Delta - \lambda)^{-1}$ is pseudodifferential and an expansion for its symbol can be found in many important cases.

2.4. A useful calculus for pseudodifferential operators has been worked out by Seeley [36]. Complex powers P^s of a pseudodifferential operator can be defined as Dunford integrals

$$P^s := \frac{i}{2\pi} \int_\Gamma \lambda^s (P - \lambda)^{-1}\, d\lambda.$$

If P is elliptic and of order $-n$, the integral kernel $k_s(x,y)$ of P^s is holomorphic in the right half-plane $\operatorname{Re} s > 1$, and $s \mapsto k_s(x,x)$ can be continued analytically to a meromorphic function with simple poles at $\{ 1 - k/n : k = 0,1,2,\dots \}$. Explicit formulas for the residues can be given in terms of the complete symbol of P.

As a consequence, the *zeta function*

$$\zeta_P(s) := \operatorname{Tr}(P^s) := \int_M k_s(x,x)\, dx \tag{2.6}$$

is a holomorphic function on the half-plane $\operatorname{Re} s > 1$, which continues analytically to a meromorphic function with the same simple poles. The residue of the zeta function at the leading pole $s = 1$ depends only on the principal symbol of P, and is given by the following formula [36, 41], which we define to be the *residue* of the pseudodifferential operator P:

$$\begin{aligned}\operatorname{Res} P &:= \operatorname*{Res}_{s=1} \zeta_P(s) = -\frac{1}{(2\pi)^{n+1} i n} \int_{\mathbb{S}^* M} \int_{\Gamma} \lambda \operatorname{tr} b_n(x,\xi,\lambda)\, d\lambda\, dx\, d\xi \\ &= \frac{1}{n(2\pi)^n} \int_{\mathbb{S}^* M} \left(-\frac{1}{2\pi i} \int_{\Gamma} \frac{\lambda}{\operatorname{tr} \sigma_{-n}(P)(x,\xi) - \lambda}\, d\lambda \right) dx\, d\xi \\ &= \frac{1}{n(2\pi)^n} \int_{\mathbb{S}^* M} \operatorname{tr} \sigma_{-n}(P)(x,\xi)\, d\xi\, dx, \end{aligned} \tag{2.7}$$

where $b_n(x,\xi,\lambda)$ denotes the principal symbol of a parametrix for the operator $P - \lambda$.

If Q is a positive elliptic pseudodifferential operator of order $-m < -n$, then its zeta function ζ_Q is holomorphic on the half-plane $\operatorname{Re} s > n/m$ and in particular at $s = 1$, so its residue vanishes; and of course one conventionally has $\sigma_{-n}(Q) = 0$. Thus we can extend the definition of $\operatorname{Res} P$ to operators of order less than $-n$ in this trivial fashion: $\operatorname{Res} P := 0$ if the order of P is less than $-n$.

We compute two simple but important examples.

Proposition 2.1. *Let Δ be the Laplacian on the n-dimensional torus $\mathbb{T}^n$ and E be the trivial line bundle on $\mathbb{T}^n$. Then*

$$n \operatorname{Res}((1-\Delta)^{-n/2}) = \Omega_n \equiv 2\pi^{n/2}/\Gamma(n/2) \tag{2.8}$$

is the area of the unit sphere $\mathbb{S}^{n-1}$.

Proof. Since $1-\Delta$ is second order, the operator $(1-\Delta)^{-n/2}$ has order $-n$, and its principal symbol is $\sigma_{-n}((1-\Delta)^{-n/2}) = \|\xi\|^{-n}$, which is the constant function 1 on the cosphere manifold $\mathbb{S}^* M$. Thus the desired residue is

$$\frac{1}{n(2\pi)^n} \int_{\mathbb{S}^* \mathbb{T}^n} 1\, d\xi\, dx = \frac{1}{n(2\pi)^n} \Omega_n \int_{\mathbb{T}^n} dx = \frac{\Omega_n}{n}. \quad \square \tag{2.9}$$

Proposition 2.2. *Let Δ be the Laplacian on the n-dimensional sphere $\mathbb{S}^n$ and E be the trivial line bundle on $\mathbb{S}^n$. Then*

$$\operatorname{Res}((1-\Delta)^{-n/2}) = 2/n!. \tag{2.10}$$

Proof. The desired residue is now

$$\frac{1}{n(2\pi)^n} \Omega_n \int_{\mathbb{S}^n} dx = \frac{\Omega_n \Omega_{n+1}}{n(2\pi)^n},$$

and the result follows from the Legendre duplication formula $\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}(n+1)) = 2^{-n+1}\sqrt{\pi}\,(n-1)!$ for the gamma function. $\square$

2.5. The deep theorem underlying the computation of the Yang–Mills functional in noncommutative geometry [8] is the surprising fact that the residue of a pseudodifferential operator of order at most $-n$ is equal to its Dixmier trace! We illustrate this first with the example of the inverse of the one-dimensional harmonic oscillator Hamiltonian $H(x,\xi) = \frac{1}{2}(x^2 + \xi^2)$. Although in this case the manifold is not compact, we still find:

$$\operatorname{Tr}^+ H^{-1} = \lim_{n\to\infty} \frac{1}{\log n} \sum_{k=0}^{n-1} \frac{2}{2k \mp 1} = 1. \tag{2.11}$$

But also

$$\operatorname*{Res}_{s=0} \operatorname{Tr}(H^{-s}) = \operatorname*{Res}_{s=1} (2^s - 1)\zeta(s) = 1, \tag{2.12}$$

where ζ is the usual Riemann zeta function; and the integral of the principal symbol is

$$\operatorname{Res} H^{-1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1 \mp x^2}\,\mathrm{d}x = 1, \tag{2.13}$$

corroborating the residue formula (2.7).

2.6. Before proving the trace theorem, we will first calculate the Dixmier trace for the previous examples.

Example 1. Let Δ be the Laplacian on the n-dimensional torus $\mathbb{T}^n$. The eigenvalues of

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

are all points l_j of the lattice $\mathbb{Z}^n$ with multiplicity one. We thus need to estimate $(\log N)^{-1} \sum_{j=1}^{N} (1 + \|l_j\|^2)^{-n/2}$ as $N \to \infty$. Let N_R be the number of lattice points in the ball of radius R centred at the origin of $\mathbb{R}^n$. Clearly $N_R \sim \operatorname{vol}\{x : \|x\| \leq R\}$, and so $N_{r+\mathrm{d}r} - N_r \sim \Omega_n r^{n-1}\,\mathrm{d}r$. If $s \leq -n/2$, we have

$$\begin{aligned} \sum_{\|l\|\leq R} (1 + \|l\|^2)^s &\sim \int_0^R (1 + r^2)^s\,(N_{r+\mathrm{d}r} - N_r) \\ &= \Omega_n \int_0^R (1 + r^2)^s r^{n-1}\,\mathrm{d}r \\ &\sim \Omega_n \int_0^R r^{2s+n-1}\,\mathrm{d}r, \end{aligned} \tag{2.14}$$

so, since $\log N_R \sim n \log R$, we get

$$(\log N_R)^{-1} \sum_{\|l\| \leq R} (1 + \|l\|^2)^s \to 0$$

for $s < -n/2$, and

$$(\log N_R)^{-1} \sum_{\|l\| \leq R} (1 + \|l\|^2)^{-n/2} \sim \frac{\Omega_n \log R}{n \log R} = \frac{\Omega_n}{n}. \tag{2.15}$$

Therefore the sequence $(\log N)^{-1} \sigma_N(P)$ converges for $P = (1 - \Delta)^{-n/2}$ and vanishes for $P = (1 - \Delta)^s$ with $s < -n/2$. Thus $(1 - \Delta)^{-n/2} \in \mathcal{L}^{1+}$ and $(1 - \Delta)^s \in \mathcal{L}_0^{1+}$ for $s < -n/2$; and moreover

$$\mathrm{Tr}^+((1 - \Delta)^{-n/2}) = \Omega_n / n. \tag{2.16}$$

Example 2. Let Δ be the Laplacian on the n-dimensional sphere $\mathbb{S}^n$. The eigenvalues [25] of Δ are $l(l + n - 1)$ with respective multiplicities $m_l = \binom{l+n}{n} - \binom{l+n-2}{n}$, for $l \in \mathbb{N}$. For example, if $n = 2$, the eigenvalues are $l(l + 1)$ and the multiplicities are $(2l + 1)$. We must estimate a_M / b_M as $M \to \infty$, where the denominator is

$$b_M := \log \sum_{l=0}^{M} m_l \sim \log\left(\binom{M + n}{n} + \binom{M + n - 1}{n}\right) \tag{2.17}$$

by telescoping, so $b_M \sim n \log M$, whereas the numerator is

$$\begin{aligned} a_M &= \sum_{l=0}^{M} m_l (1 + l(l + n - 1))^{-n/2} \sim \sum_{l=0}^{M} \frac{2\binom{l+n-1}{n-1}}{(l + \frac{1}{2}n)^n} \\ &\sim \frac{2}{(n - 1)!} \sum_{l=0}^{M} (l + \tfrac{1}{2}n)^{-1} \sim \frac{2 \log M}{(n - 1)!}. \end{aligned} \tag{2.18}$$

Thus

$$\mathrm{Tr}^+((1 - \Delta)^{-n/2}) = \lim_{M \to \infty} \frac{2 \log M / (n - 1)!}{n \log M} = \frac{2}{n!}. \tag{2.19}$$

Moreover, if we replace the exponent $-n/2$ by a lesser s, the series for a_M becomes convergent, and so the Dixmier trace vanishes, just as in the former example.

2.7. In both examples, for $P = (1 - \Delta)^{-n/2}$, we find that $\mathrm{Tr}^+ P = \mathrm{Res}\, P$. Connes' trace theorem [8] is that this equality holds in general.

The proof originally given by Connes consists in showing that the Dixmier trace equals the normalized integral of the principal symbol, i.e., the right hand side of the definition (2.7). This is done by showing that the theorem may be reduced to one special case, namely $P = (1 - \Delta)^{-n/2}$ on $M = \mathbb{S}^n$, where equality follows by comparing (2.10) and (2.19). Because of the central rôle of this theorem, we give here a different proof.

Theorem 2.3. *Let M be a compact Riemannian manifold of dimension n, E a complex vector bundle on M, and $P: \Gamma(E) \to \Gamma(E)$ a pseudodifferential operator of order $-n$. Then $P \in \mathcal{L}^{1+}(L^2(E))$ and*

$$\operatorname{Tr}^+(P) = \operatorname{Res} P = \frac{1}{n\,(2\pi)^n} \int_{\mathbb{S}^* M} \operatorname{tr} \sigma_{-n}(P). \tag{2.20}$$

Moreover, this quantity depends only on the conformal class of the metric on M.

Proof. The Hilbert space $\mathcal{H}$ on which P acts is the completion of $\Gamma(E)$ to the space of square-integrable sections $L^2(E)$ with respect to the inner product obtained from the Riemannian metric. If $\mathcal{H}_1$ and $\mathcal{H}_2$ come from two conformally equivalent metrics, the identity operator on $\Gamma(E)$ extends to a linear map $T: \mathcal{H}_1 \to \mathcal{H}_2$ which is bounded with bounded inverse. Since $\operatorname{Tr}^+(TPT^{-1}) = \operatorname{Tr}^+(P)$, the Dixmier ideal and traces are the same for both. Furthermore, the cosphere bundle $\mathbb{S}^*M$, as a submanifold of T^*M, depends on a choice of metric in M; but since $\sigma_{-n}(P)$ is homogeneous of degree $-n$ in ξ, the change-of-variables formula shows that the integral on the right of (2.20) is constant within each conformal class.

By linearity of $\operatorname{Tr}^+ P$ and $\sigma_n(P)$ in P, it suffices to establish the theorem for P a compact positive operator. In view of the equality (2.7), it suffices to show that $\operatorname{Tr}^+ P$ is finite and equals $\operatorname{Res} P = \operatorname{Res}_{s=1} \zeta_P(s) = \operatorname{Res}_{s=1} \sum_{k=0}^{\infty} s_k(P)^s$ for P pseudodifferential of order $-n$.

The residue of the zeta function of a positive operator may be related to the size of its eigenvalues by using the Ikehara–Wiener Tauberian theorem. We recall [26] that this can be stated as follows. Suppose that $f(s) = \int_0^\infty e^{-st} F(t)\,dt$ is a Laplace transform, where $F: [0, +\infty) \to [0, +\infty)$ is a piecewise continuous nondecreasing function with at most jump discontinuities at an isolated set of points, that $f(s)$ is analytic in a half-plane $\operatorname{Re} s > s_0 > 0$ and extends to be analytic on $\operatorname{Re} s \geq s_0$ save for a simple pole at $s = s_0$ with a positive residue $C > 0$. Then the Ikehara–Wiener theorem assures us that

$$F(t) \sim C\, e^{s_0 t}, \quad \text{as } t \to \infty. \tag{2.21}$$

Now it suffices to consider the nondecreasing step function

$$F(t) := \sum_{k=0}^{\infty} \theta(t + \log s_{k+k_0}(P)), \tag{2.22}$$

where θ denotes the Heaviside function and $s_{k_0}(P)$ is the first eigenvalue of P less than 1. Since the Laplace transform of $\theta(t-a)$ is $s^{-1}e^{-as}$ for $a > 0$, we find that

$$f(s) = \frac{1}{s} \sum_{k=0}^{\infty} s_{k+k_0}(P)^s = \frac{1}{s}\Big(\zeta_P(s) - \sum_{k<k_0} s_k(P)^s\Big). \tag{2.23}$$

Since P is pseudodifferential of order $-n$, we know that ζ_P and therefore f satisfies the hypotheses of the Ikehara–Wiener theorem with $s_0 = 1$; and we see that

$$C = \operatorname*{Res}_{s=1} f(s) = \operatorname*{Res}_{s=1} \zeta_P(s) = \operatorname{Res} P. \tag{2.24}$$

From (2.21), we conclude that $F(t) \sim C\,e^t$ as $t \to \infty$.

However, (2.22) may be rewritten as

$$F(t) = k \iff s_{k+k_0}(P)^{-1} \le e^t < s_{k+k_0+1}(P)^{-1}, \tag{2.25}$$

from which it follows that $s_{k+k_0}(P) \sim C/k$ as $k \to \infty$. We have at once that $P \in \mathcal{L}^{1+}(\mathcal{H})$ and that

$$\operatorname{Tr}^+ P = C = \operatorname{Res} P. \qquad \square$$

Corollary 2.4 (Weyl's theorem on spectral asymptotics for the Laplacian). *On a compact n-dimensional Riemannian manifold, the eigenvalues λ_m of the Laplacian satisfy*

$$\lambda_m(-\Delta) \sim 4\pi^2 \Big(\frac{n}{\operatorname{vol} M}\Big)^{2/n} \Big(\frac{m}{\Omega_n}\Big)^{2/n}. \tag{2.26}$$

Proof. The finiteness of $\operatorname{Tr}^+(1-\Delta)^{-n/2}$ gives $\lambda_m(-\Delta) = O(m^{2/n})$. The residue formula relates this to the volume of the manifold, since the principal symbol of $(1-\Delta)$ is the constant 1 on $\mathbb{S}^*M$. Indeed, $\lambda_m(-\Delta)^{-n/2} \sim C/m$, where $C = \operatorname{Res} P = \Omega_n \operatorname{vol} M/n\,(2\pi)^n$. $\square$

Thus Connes' trace theorem places Weyl's theorem in a new light, as a harbinger of the theory of noncommutative integration. It is worth mentioning that the pursuit of this golden thread led Guillemin to study the residue [22]. Connes' use of the Dixmier trace provides the eigenvalue estimates needed to link geometry with analysis.

2.8. For completeness, we expound the original proof of Connes of the trace theorem. (The proof in ref. [8] contains a few misprints.) We have already established the special case $M = \mathbb{S}^n$, $P = (1-\Delta)^{-n/2}$, and must show the general case reduces to this one. We take (2.7) as known; its right hand side is finite for pseudodifferential operators of order at most $-n$ and vanishes on operators of order less than $-n$. Thus $\operatorname{Res} P$ defines a trace on this class of operators. It is therefore enough to show that these operators lie in $\mathcal{L}^{1+}(\mathcal{H})$ and that any two traces are proportional on this class.

We can write P as a finite sum of operators of the type $s \mapsto \phi P(\psi s)$, where ϕ, ψ belong to partitions of unity of M. Multiplication operators are bounded on $\mathcal{H}$, so membership in the ideals $\mathcal{L}^{1+}$ is unaffected by $P \mapsto \phi P \psi$, and we can restrict to one coordinate chart, supposing that M is flat and that E is a trivial bundle. Alternatively, we can suppose that M is a given n-dimensional compact manifold; we take $M = \mathbb{S}^n$.

Now $S = P(1-\Delta)^{n/2}$ is an operator of order 0, and thus is bounded. Thus $P = S(1-\Delta)^{-n/2}$ with S bounded. Since $\mathcal{L}^{1+}(\mathcal{H})$ is an ideal, and $(1-\Delta)^{-n/2} \in \mathcal{L}^{1+}(\mathcal{H})$ by (2.19), we obtain $P \in \mathcal{L}^{1+}(\mathcal{H})$. Since we also know that $(1-\Delta)^{s/2} \in \mathcal{L}_0^{1+}$ for $s < -n$, the same argument shows that any pseudodifferential operator on M of order $s < -n$ lies in $\mathcal{L}_0^{1+}$ and so its Dixmier trace vanishes. This is in particular true for the operator of order $(-n-1)$ whose complete symbol is $p(x,\xi) - p_{-n}(x,\xi)$. Thus $\mathrm{Tr}^+(P)$ depends only on the principal symbol of P.

The space of all $\mathrm{tr}\, \sigma_{-n}(P)$ is just $C^\infty(\mathbb{S}^*M)$ (any smooth function on $\mathbb{S}^*M$ is of this form [31]); and $\sigma_{-n}(P) \mapsto \mathrm{Tr}^+(P)$ is a continuous linear form on this space, i.e., a distribution on the compact manifold $\mathbb{S}^*M$. Now the Dixmier trace is a *positive* linear functional, and nonnegative principal symbols correspond to positive operators; so this distribution is positive. A positive distribution is a measure m [17], so $\mathrm{Tr}^+(P) = \int_{\mathbb{S}^*M} \sigma_{-n}(P)\, \mathrm{d}m(x,\xi)$.

An isometry $\phi: \mathbb{S}^n \to \mathbb{S}^n$ transforms $\sigma_{-n}(P)(x,\xi)$ to $\sigma_{-n}(P)(\phi(x), \phi^*\xi)$ in local coordinates (where ϕ^* is the transpose of the Jacobian of ϕ), and determines a unitary operator U_ϕ on $\mathcal{H}$; P transforms to $U_\phi P U_\phi^{-1}$. Since $\mathrm{Tr}^+(U_\phi P U_\phi^{-1}) = \mathrm{Tr}^+(P)$, the measure m determined by Tr^+ is invariant under all such isometries. In particular, if ϕ is a rotation in $\mathrm{SO}(n+1)$ and if $\phi(x) = x$, then $\xi \mapsto \phi^*\xi$ on $\mathbb{S}_x^*\mathbb{S}^n$ is the transpose of the corresponding rotation in the isotropy subgroup $\mathrm{SO}(n)$ of x; in other words, $\mathbb{S}^*\mathbb{S}^n$ is a homogeneous space for the action of $\mathrm{SO}(n+1)$. Now any $\mathrm{SO}(n+1)$-invariant measure is proportional to the volume form on $\mathbb{S}^*\mathbb{S}^n$. We therefore have:

$$\mathrm{Tr}^+(P) \propto \frac{1}{n(2\pi)^n} \int_{\mathbb{S}^*\mathbb{S}^n} \mathrm{tr}\, \sigma_{-n}(P)\, \mathrm{d}\xi\, \mathrm{d}x = \mathrm{Res}\, P. \tag{2.27}$$

The foregoing argument actually shows that any trace which can be expressed as a function of the principal symbol is proportional to $\mathrm{Res}\, P$, so it may be applied to any of the various Tr_ω introduced by Dixmier. Now, however, from (2.10) and (2.19), the proportionality constant is 1, so $\mathrm{Tr}^+ P$ is unambiguous and equals $\mathrm{Res}\, P$, as claimed. □

3. Noncommutative geometry: algebras and modules

3.1. A compact topological space M gives naturally rise to a commutative C^*-algebra, to wit, the algebra $C(M)$ of complex continuous functions defined on M with the sup norm. One can distinguish topologies on M by the properties of $C(M)$. In fact, much more is true: given any nontrivial commutative C^*-

algebra $\mathcal{A}$ with identity, the Gelfand–Naĭmark theorem produces a compact topological space M, which can be reconstructed from $\mathcal{A}$, such that $\mathcal{A} = C(M)$.

A *character* of a complex Banach algebra $\mathcal{A}$ is a nonzero homomorphism $\mu: \mathcal{A} \to \mathbb{C}$. We denote by $M(\mathcal{A})$ the set of characters of $\mathcal{A}$. We may assume that $\mathcal{A}$ has an identity 1; otherwise one can adjoin an external one and use the augmented algebra $\mathcal{A}^+ := \mathbb{C} \times \mathcal{A}$. The characters of a commutative unital Banach algebra form a nonempty closed subset of the unit ball of the dual space $\mathcal{A}'$ (with the weak *-topology), and as such is a compact set, by Alaoglu's theorem. This "Gelfand topology" of $M(\mathcal{A})$ is the weakest topology which makes all evaluation maps $\mu \mapsto \mu(a)$ continuous; thus if we write $\hat{a}(\mu) := \mu(a)$, the Gelfand transform $\hat{a}$ belongs to $C(M(\mathcal{A}))$. The Gelfand–Naĭmark theorem then assures us that for $\mathcal{A}$ a commutative unital C^*-algebra, the homomorphism $a \mapsto \hat{a}$ is in fact an isometric $*$-isomorphism of $\mathcal{A}$ onto $C(M(\mathcal{A}))$.

Thus, not only is it true that, given a compact, Hausdorff topological space M, there is the algebra $C(M)$ associated to it; it is also true that a given commutative unital C^*-algebra is associated to only one topological space, which may be reconstructed from the algebra, to wit, the character space. Beyond its future uses, the theorem gives us confidence to think of general C^*-algebras as noncommutative topological spaces.

Now, to some extent, algebras of smooth functions can be substituted for algebras of continuous functions. Consider a compact smooth manifold and $\mathcal{A} := C^\infty(M)$. Then $M(\mathcal{A}) = M$. The last equality extends the Gelfand–Naĭmark theorem and can be seen as follows. First, a character μ of the symmetric Fréchet algebra $C^\infty(M)$ is a multiplicative linear form, and as such is automatically continuous [34]; hence it is a distribution on M. Since $\mu(|a|^2) = \mu(a^*a) = \mu(a^*)\mu(a) = |\mu(a)|^2 \geq 0$, this is a positive distribution and thus is a *measure* on M [17]. That is to say, it extends to a continuous character on the C^*-algebra $C(M)$; so we conclude that $M(\mathcal{A}) = M$. Unfortunately, there seems to be no easy way to characterize algebras isomorphic to $C^\infty(M)$ among commutative involutive Fréchet algebras.

The pair $(C^\infty(M), C(M))$ is a commutative "smooth algebra". We have just seen that the first element of the pair determines the second; for a general smooth algebra $\mathcal{A} := (\mathcal{A}^\infty, \overline{\mathcal{A}^\infty})$, where $\overline{\mathcal{A}^\infty}$ may be a noncommutative C^*-algebra and $\mathcal{A}^\infty$ is an involutive Fréchet subalgebra dense in $\overline{\mathcal{A}^\infty}$, the situation is more involved [40]. The program of noncommutative geometry is to adapt the classical tools for dealing with the manifold M, such as K-theory, de Rham cohomology, ..., to the case where the pair $(C^\infty(M), C(M))$ is replaced by a noncommutative smooth algebra.

3.2. There is also an algebraic characterization of vector bundles [27, 38]. Given a manifold M, we shall consider a (real or complex) vector bundle $E \to M$; we shall denote by $\Gamma(E)$ the space of smooth sections. We view it as a right $\mathcal{A}$-

module in the obvious way.

Definition. A right module P for an algebra $\mathcal{A}$ is called *projective* if it has the following lifting property: given a surjective homomorphism $\pi: B \to C$ of right $\mathcal{A}$-modules, any homomorphism $f: P \to C$ can be lifted to a homomorphism $\tilde{f}: P \to B$ with $\pi \circ \tilde{f} = f$. This is clearly the case for a free module (just define $\tilde{f}$ on a set of generators), and also whenever there is another module Q such that $P \oplus Q$ is free. The latter turns out to be the general case: a module is projective iff it is a direct summand of a free module. (Any module is the quotient of some free module F, and the projective property allows us to split the quotient map $\epsilon: F \to P$; i.e., there will exist $\kappa: P \to F$ such that $\epsilon \circ \kappa = \mathrm{id}_P$.) We say P is of *finite type* if it is finitely generated. A projective module of finite type (*finite projective* for short) is then an $\mathcal{A}$-module P for which there is an integer k and another module Q so that $P \oplus Q \simeq \mathcal{A}^k$.

We note that if $\mathcal{A}$ has an appropriate topology, the quotient map $\epsilon: \mathcal{A}^k \to P$ confers a natural topology on the finite projective module P.

Proposition 3.1. *The $C^\infty(M)$-module $\Gamma(E)$ is finite projective.*

Proof. It is clear that if E is a trivial vector bundle of rank k, so that $E \simeq M \times \mathbb{C}^k$ (for a complex vector bundle, say), then $\Gamma(E)$ is just the free module $\mathcal{A}^k$. In general, it is enough to identify an integer n and maps $\kappa: \Gamma(E) \to \mathcal{A}^n$, $\epsilon: \mathcal{A}^n \to \Gamma(E)$ so that $\epsilon \circ \kappa = \mathrm{id}_{\Gamma(E)}$. Then the image of the idempotent $p = \kappa \circ \epsilon$ is the desired direct summand. Let U_i, $1 \le i \le q$, be an open trivializing covering of the compact manifold M; we can represent an element of $\Gamma(E)$ by q differentiable maps $s_i: U_i \to \mathbb{C}^k$ satisfying the compatibility conditions

$$s_j(x) = \sum_i g_{ji}(x) s_i(x) \quad \text{on } U_i \cap U_j \tag{3.1}$$

(where the g_{ji} are the transition functions of E) and the obvious module operation.

Let h_i, $1 \le i \le q$, denote a partition of unity subordinate to the covering $\{U_i\}$. Set $l_i := h_i(h_1^2 + \cdots + h_q^2)^{-1/2}$; then l_i^2, $1 \le i \le q$, is also a partition of unity subordinate to $\{U_i\}$. Set now $n = kq$ and view R^n as $R^k \oplus \cdots \oplus R^k$ (q summands). Define κ, ϵ by

$$\begin{aligned} \kappa(s_1, \dots, s_q) &:= (l_1 s_1, \dots, l_q s_q), \\ \epsilon(t_1, \dots, t_q) &:= (\tilde{s}_1, \dots, \tilde{s}_q) \quad \text{with } \tilde{s}_i = \textstyle\sum_r g_{ir} l_r t_r. \end{aligned} \tag{3.2}$$

Clearly, $\epsilon \circ \kappa = \mathrm{id}_{\Gamma(E)}$. Note that compactness is decisive for this argument. □

The converse problem is to reconstruct, from a given finite projective module, a vector bundle whose module of sections coincides with the given one. This reconstruction is effected by the Serre–Swan theorem [38].

Theorem 3.2 (Serre–Swan). *An $C^\infty(M)$-module P is isomorphic to a module of the form $\Gamma(E)$ iff it is finite projective.*

Proof. The "only if" part is proposition 3.1. Suppose now that P is finite projective, i.e., a direct summand of a free, finitely generated $C^\infty(M)$-module F. There will be an idempotent endomorphism $f: F \to F$ with $P = \operatorname{im} f$. Now if k is the dimension of F, we have $F = \Gamma(E)$, where E is a trivial vector bundle of rank k.

Since f is a module map, we have $f(sh) = f(s)h$ for $h \in C^\infty(M)$. If $x \in M$, and I_x is the ideal of functions $h \in C^\infty(M)$ such that $h(x) = 0$, then f preserves the submodule $\Gamma(E)I_x$. Since $s \mapsto s(x)$ induces a linear isomorphism of $\Gamma(E)/\Gamma(E)I_x$ onto the fibre E_x, we have $f(s)(x) \in E_x$ for all $s \in \Gamma(E)$, and $p: E \to E : s(x) \mapsto f(s)(x)$ defines a bundle homomorphism satisfying $f(s) = p \circ s$. Since $f^2 = f$, we clearly have $p^2 = p$.

If $\dim p(E_x) = r$, then we can find r linearly independent smooth local sections $s_1, \dots, s_r$ of $E \to M$ near $x \in M$ such that $ps_j(x) = s_j(x)$ for each j. Hence, $ps_1, \dots, ps_r$ are linearly independent in a neighbourhood U of x, so $\dim p(E_y) \geq r$ for $y \in U$. The same argument applied to the idempotent $(1-p): E \to E$ shows that $\dim(1-p)(E_y) \geq k - r$ for y near x. Thus $x \mapsto \dim p(E_x)$ is locally constant, and so $E' = p(E)$ is the total space of a vector bundle $E' \to M$ with fibres $p(E_x)$, for which $E = E' \oplus \ker p$. From the definition of E', one sees that $\Gamma(E') = \{p \circ s : s \in \Gamma(E)\} = \operatorname{im} f = P$. □

3.3. The Gelfand–Naĭmark theorem indicates that a differentiable manifold M is described entirely by the commutative $*$-algebra $C^\infty(M)$. In general, a noncommutative space will be given by a general smooth algebra $\mathcal{A}$. We shall work with unital algebras whenever feasible.

We shall be using systematically two paradigmatic examples, which are of direct interest for the application to particle physics: the example of a compact smooth (even-dimensional, spin^c) manifold, where $\mathcal{A} := C^\infty(M)$ and $M(\mathcal{A}) = M$, and the following:

Example. A two-point set, which can described by the algebra $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$, with the usual operations working componentwise. We have $M = \{q_1, q_2\}$, where $q_1(z_1, z_2) = z_1$ and $q_2(z_1, z_2) = z_2$.

3.4. A right (left) $\mathcal{A}$-module $\mathcal{E}$ is a complex vector space on which $\mathcal{A}$ acts on the right (left). When tensor-multiplying modules over $\mathcal{A}$, we shall multiply right modules by bimodules in order to get right modules, bimodules by left modules in order to get left modules, and so on. For instance, the tensor product of a right module $\mathcal{E}_1$ by a bimodule $\mathcal{E}_2$ is a right module $\mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2$ generated by the set $\{ s_1 \otimes s_2 : s_1 \in \mathcal{E}_1,\ s_2 \in \mathcal{E}_2 \}$ with the relations

$$s_1 a \otimes s_2 - s_1 \otimes a s_2 = 0 \quad \text{for any } a \in \mathcal{A}. \tag{3.3}$$

A remark on notation: the symbol $\otimes$, when employed to denote tensor product of spaces, will mean the usual tensor product over the complex numbers; we write $\otimes_{\mathcal{A}}$ to denote product over an algebra $\mathcal{A}$, in order to avoid any confusion.

If $\mathcal{E}_1$, $\mathcal{E}_2$ are, say, right modules, we shall consider $\mathrm{End}_{\mathcal{A}}(\mathcal{E}_1, \mathcal{E}_2)$, the space of $\mathbb{C}$-linear maps Λ from $\mathcal{E}_1$ into $\mathcal{E}_2$ satisfying $\Lambda(sa) = \Lambda(s)a$ for $a \in \mathcal{A}$, $s \in \mathcal{E}_1$. (We shall speak of "$\mathcal{A}$-linear maps" in this case.) In particular, $\mathrm{End}_{\mathcal{A}}(\mathcal{E}) := \mathrm{End}_{\mathcal{A}}(\mathcal{E}, \mathcal{E})$ and $\mathcal{E}' := \mathrm{End}_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$.

Recall that $\mathcal{E}$ is a finite projective right module over $\mathcal{A}$ iff there exist $m \in \mathbb{N}$ and $p \in \mathcal{A}^{m \times m}$, such that $p^2 = p$ and $\mathcal{E} = p\mathcal{A}^m$. In view of the Serre–Swan theorem, we shall call such a module a *vector bundle* over $\mathcal{A}$. If p can be taken to be the identity, then $\mathcal{E}$ is a trivial vector bundle. Note that the elements of $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$ are matrices $v \in \mathcal{A}^{m \times m}$ satisfying $pv = vp$.

Example 1. Consider the space of smooth sections $\mathcal{E} = \Gamma(E)$ of a smooth finite-dimensional vector bundle E over a compact smooth (even-dimensional, spin^c) manifold M.

Example 2. Take $\mathcal{E} = \mathbb{C}^k \oplus \mathbb{C}^l$. Note that if $k \neq l$, then $\mathcal{E}$ is not trivial. (If $k > l$, take $p = 1_k \oplus p' \in (\mathbb{C} \oplus \mathbb{C}) \otimes \mathbb{C}^{k \times k} = \mathbb{C}^{k \times k} \oplus \mathbb{C}^{k \times k}$, where $p' = 1_l \oplus 0_{k-l}$.)

3.5. **Definition.** We shall consider *hermitian vector bundles*. This means that we have a sesquilinear map $(\cdot \mid \cdot) : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ such that:

(i) $(sa \mid tb) = a^*(s \mid t)b$ for $s, t \in \mathcal{E}$, $a, b \in \mathcal{A}$;
(ii) $(s \mid t)^* = (t \mid s)$ in $\mathcal{A}$ for $s, t \in \mathcal{E}$;
(iii) $(s \mid s) \in \mathcal{A}$ is positive for all $s \in \mathcal{E}$ and $(s \mid s) = 0$ forces $s = 0$.

Proposition 3.3. *Hermitian vector bundles are of the form $p\mathcal{A}^m$ with p self-adjoint.*

Proof. There is an obvious hermitian structure on the trivial vector bundle $\mathcal{A}^m$: if $s = (a_1, \ldots, a_n)$, $t = (b_1, \ldots, b_n)$, define $(s \mid t) := \sum_{j=1}^n a_j^* b_j$. If $\mathcal{E} = p\mathcal{A}^m$ is a submodule of $\mathcal{A}^m$, let $\mathcal{E}^\perp := \{ u \in \mathcal{A}^m : (u \mid s) = 0 \text{ for } s \in \mathcal{E} \}$. Since $(ua \mid s) = a^*(u \mid s)$, $\mathcal{E}^\perp$ is also an $\mathcal{A}$-module. If $u \in \mathcal{A}^m$, then $(t - p^*t \mid s) = (t \mid s - ps)$, so we see that $\mathcal{E}^\perp = (1 - p^*)\mathcal{A}^m$. Since $\mathcal{A}^m = p\mathcal{A}^m \oplus (1 - p)\mathcal{A}^m$ as a direct sum of modules, the operation $(\cdot \mid \cdot)$ restricts to a hermitian structure on $p\mathcal{A}^m$ iff this is an orthogonal direct sum, i.e., iff $1 - p^* = 1 - p$.

We have used the fact that $(p^*t \mid s) = (t \mid ps)$ for $s, t \in \mathcal{A}^m$ and $p \in \mathcal{A}^{m\times m}$. To check this, notice that $ps = \left(\sum_j p_1^j a_j, \dots, \sum_j p_m^j a_j\right)$, and that $p^*t = \left(\sum_k (p_k^1)^* b_k, \dots, \sum_k (p_k^m)^* b_k\right)$, so the desired identity reduces to the usual matrix calculation. □

The unitary group $\mathcal{U}_m(\mathcal{A})$ of a $*$-algebra $\mathcal{A}$ is $\{\, u \in \mathcal{A}^{m\times m} : uu^* = u^*u = 1 \,\}$. More generally, for hermitian vector bundles we can consider the group $\mathcal{U}(\mathcal{E})$ of *gauge transformations* of $\mathcal{E}$, given by $\{\, u \in \operatorname{End}_{\mathcal{A}}(\mathcal{E}) : uu^* = u^*u = 1 \,\}$.

Example 1. It is clear that Hilbert structures over vector bundles give rise to hermitian forms over the space of sections. It is also clear that $\mathcal{U}_m(C^\infty(M))$ is isomorphic to the group of maps $C^\infty(M, \mathrm{U}(m))$. More generally, it is seen that our group of gauge transformations is identified with the group of gauge transformations "of the second kind" for the bundle with structure group $\mathrm{U}(m)$, in the usual sense.

Example 2. We can take $((s_1, s_2) \mid (t_1, t_2)) = (s_1^* t_1, s_2^* t_2)$. The group of gauge transformations is $\mathrm{U}(k) \times \mathrm{U}(l)$.

4. Noncommutative geometry: forms and connections

4.1. We now want to find a substitute for the de Rham complex in noncommutative geometry. We approach this *par le biais* of the following universal problem [7, 28]: Given any derivation D of the unital algebra $\mathcal{A}$ into a bimodule $\mathcal{E}$, find an injection and a derivation d of $\mathcal{A}$ into a (graded) differential algebra $\Omega^\bullet\mathcal{A}$ such that there is a unique bimodule morphism $\phi\colon \Omega^\bullet\mathcal{A} \to \mathcal{E}$ with $D = \phi \circ d$. Also, any homomorphism of $\mathcal{A}$ into the degree zero subalgebra of a differential algebra must be lifted to a homomorphism of differential algebras.

It is plausible that such an algebra can be thought of as a free algebra generated by symbols $\{\, a, da : a \in \mathcal{A} \,\}$ subject to the relations $d1 = 0$, $d(a_0 a_1) - da_0\, a_1 - a_0\, da_1 = 0$. The latter would allow one to write elements of $\Omega\mathcal{A}$ as linear combinations of monomials of the form $a_0\, da_1 \cdots da_n$. The relations

$$d(a_0\, da_1 \cdots da_n) = da_0\, da_1 \cdots da_n, \qquad d(da_0\, da_1 \cdots da_n) = 0 \tag{4.1}$$

are equivalent to the requirement that $d^2 = 0$.

4.2. The actual construction of $\Omega^\bullet\mathcal{A}$ turns out to follow rather conventional lines [4]. All that is required is a stomach which can resist a starchy diet of tensor products.

Introduce $C_n(\mathcal{A}) := \mathcal{A}^{\otimes(n+1)}$. We write $C_\bullet(\mathcal{A}) = \mathcal{A}^{\otimes\bullet+1}$ for the chain complex. We have a boundary operator

$$b'(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := \sum_{i=0}^{n-1} (-)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n. \tag{4.2}$$

For example, $b'(a_0) = 0$, $b'(a_0 \otimes a_1) = a_0 a_1$, $b'(a_0 \otimes a_1 \otimes a_2) = a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2$. Notice that $b': C_1(\mathcal{A}) \to C_0(\mathcal{A})$ is just the multiplication map $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$.

A short calculation verifies that $b'^2 = 0$. As usual, we introduce the spaces of boundaries $B_n^{\mathrm{a}}(\mathcal{A})$ and the set of cycles $Z_n^{\mathrm{a}}(\mathcal{A})$. We recall that a complex is called "acyclic" (indicated here by the superscript a) if all the homology groups are zero, with the possible exception of H_0. It can be seen that the homology of $(C_\bullet(\mathcal{A}), b')$ is entirely trivial by considering the map of degree 1:

$$s(a_0 \otimes \cdots \otimes a_n) := 1 \otimes a_0 \otimes \cdots \otimes a_n, \tag{4.3}$$

and computing

$$\begin{aligned} b's(a_0 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_n \\ &\quad + \sum_{i=0}^{n-1} (-)^{i+1} 1 \otimes a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n, \\ sb'(a_0 \otimes \cdots \otimes a_n) &= \sum_{i=0}^{n-1} (-)^i 1 \otimes a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n, \end{aligned} \tag{4.4}$$

so $b's + sb' = \mathrm{id}$; i.e., s is a "chain homotopy" between id and 0, and $(C_\bullet(\mathcal{A}), b')$ is acyclic. Explicitly: if $b'\alpha = 0$, then $\alpha = b'(s\alpha)$, so every n-cocycle is an n-coboundary.

4.3. We now introduce the $\mathcal{A}$-bimodule

$$\begin{aligned} \Omega^1\mathcal{A} &:= \ker(m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}) \\ &= \ker(b': C_1(\mathcal{A}) \to C_0(\mathcal{A})) \\ &= Z_1^{\mathrm{a}}(\mathcal{A}) = B_1^{\mathrm{a}}(\mathcal{A}). \end{aligned} \tag{4.5}$$

The fact that $\Omega^1\mathcal{A} = Z_1^{\mathrm{a}}(\mathcal{A}) = B_1^{\mathrm{a}}(\mathcal{A})$ means that the following two sequences are exact:

$$\begin{gathered} 0 \longrightarrow \Omega^1\mathcal{A} \xrightarrow{\ i\ } \mathcal{A} \otimes \mathcal{A} \xrightarrow{\ m\ } \mathcal{A} \longrightarrow 0, \\ \mathcal{A}^{\otimes 4} \xrightarrow{\ b'\ } \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \xrightarrow{\ j\ } \Omega^1\mathcal{A} \longrightarrow 0. \end{gathered} \tag{4.6}$$

Here the injection i and the surjection j give the canonical factorization of the map $b' = m \otimes \mathrm{id} - \mathrm{id} \otimes m: \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, i.e., $ij = m \otimes \mathrm{id} - \mathrm{id} \otimes m$. Thus $\Omega^1\mathcal{A}$ is the cokernel of $b': C_3(\mathcal{A}) \to C_2(\mathcal{A})$, so any bimodule homomorphism $f: \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to M$ such that $fb' = 0$ factors through j, i.e., $f = f'j$ for a unique $f': \Omega^1\mathcal{A} \to M$.

Define next $d: \mathcal{A} \to \Omega^1\mathcal{A}$ by $da := 1 \otimes a - a \otimes 1$. Now $\Omega^1\mathcal{A}$ is an $\mathcal{A}$-bimodule under the obvious recipe: $a'(\sum_k x_k \otimes y_k)a'' := \sum_k a'x_k \otimes y_k a''$. We also have

$$d(ab) = 1 \otimes ab - ab \otimes 1 = a \otimes b - ab \otimes 1 + 1 \otimes ab - a \otimes b = a\,db + da\,b. \quad (4.7)$$

Clearly $d1 = 0$. Thus $d: \mathcal{A} \to \Omega^1\mathcal{A}$ is a derivation.

Now suppose $\mathcal{E}$ is any $\mathcal{A}$-bimodule and let $D: \mathcal{A} \to \mathcal{E}$ be a derivation, i.e., an additive map which kills 1 and verifies $D(ab) = Da\,b + a\,Db$. Define an $\mathcal{A}$-bimodule homomorphism $\tilde{D}: \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to \mathcal{E} : a \otimes b \otimes c \mapsto a\,Db\,c$. We compute:

$$\begin{aligned} &\tilde{D}b'(a_0 \otimes a_1 \otimes a_2 \otimes a_3) \\ &\quad = \tilde{D}(a_0a_1 \otimes a_2 \otimes a_3 - a_0 \otimes a_1a_2 \otimes a_3 + a_0 \otimes a_1 \otimes a_2a_3) \\ &\quad = a_0(a_1\,Da_2 - D(a_1a_2) + Da_1\,a_2)a_3 = 0. \end{aligned} \quad (4.8)$$

Thus there is a unique $\mathcal{A}$-bimodule map $\phi: \Omega^1\mathcal{A} \to \mathcal{E}$ with $\tilde{D} = \phi j$. So we have that $\phi(a_0a_1 \otimes a_2 - a_0 \otimes a_1a_2) = a_0\,Da_1\,a_2$; hence $(-\phi)(da) = Da$. We have thus established the universality of $(\Omega^1\mathcal{A}, d)$.

4.4. We now wish to extend the derivation $d: \mathcal{A} \to \Omega^1\mathcal{A}$ to a differential graded algebra $\Omega^\bullet\mathcal{A}$ with $\Omega^0\mathcal{A} := \mathcal{A}$ with an additive differential $d: \Omega^n\mathcal{A} \to \Omega^{n+1}\mathcal{A}$, commutative in the standard graded sense:

$$d(\alpha\beta) = d\alpha\,\beta + (-)^{\deg\alpha}\alpha\,d\beta \quad (4.9)$$

for $\alpha \in \Omega^n(\mathcal{A})$, $\beta \in \Omega^\bullet\mathcal{A}$, the case $n = 0$ being the original derivation property of d; and such that $d^2 = 0$.

This algebra $\Omega^\bullet\mathcal{A}$ should be "universal" in the following sense: if $(R^\bullet, \delta)$ is another differential graded algebra, any algebra homomorphism $\psi: \mathcal{A} \to R^0$ should extend to an algebra homomorphism of degree zero $\psi: \Omega^\bullet\mathcal{A} \to R^\bullet$ intertwining the differentials d and δ. Of course, the algebra product in $\mathcal{A}$ and the derivation d must determine the product in $\Omega^\bullet\mathcal{A}$.

Introduce $\overline{\mathcal{A}} := \mathcal{A}/\mathbb{C}$. Then we remark that $\Omega^1\mathcal{A} = \mathcal{A} \otimes \overline{\mathcal{A}}$ by the identification $a_0 \otimes \bar{a}_1 \mapsto a_0\,da_1$. Here $\bar{a}_1 = a_1 + \mathbb{C}1$ in $\overline{\mathcal{A}}$, but of course da_1 is unambiguous since $d1 = 0$. If $c \in \mathcal{A}$, then $c(a_0 \otimes \bar{a}_1) \mapsto ca_0\,da_1$, while

$$\begin{aligned} (a_0 \otimes \bar{a}_1)c &=: a_0 \otimes (a_1c)^- - a_0a_1 \otimes \bar{c} \\ &\longmapsto a_0\,d(a_1c) - a_0a_1\,dc = a_0\,da_1\,c, \end{aligned} \quad (4.10)$$

so this correspondence is a bimodule isomorphism.

Next we put $\Omega^2\mathcal{A} := \Omega^1\mathcal{A} \otimes_{\mathcal{A}} \Omega^1\mathcal{A} = (\mathcal{A} \otimes \overline{\mathcal{A}}) \otimes_{\mathcal{A}} (\mathcal{A} \otimes \overline{\mathcal{A}}) = \mathcal{A} \otimes \overline{\mathcal{A}} \otimes \overline{\mathcal{A}}$. More generally, we define $\Omega^n\mathcal{A} := \Omega^1\mathcal{A} \otimes_{\mathcal{A}} \Omega^1\mathcal{A} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \Omega^1\mathcal{A}$ (n times), so that $\Omega^n\mathcal{A} = \mathcal{A} \otimes \overline{\mathcal{A}}^{\otimes n}$. In other words, we take the tensor algebra over $\mathcal{A}$, but we quotient out the scalar terms except in degree zero; this is what makes $d^2 = 0$, yielding a graded differential algebra.

The differential $d: \mathcal{A} \otimes \overline{\mathcal{A}}^{\otimes n} \to \mathcal{A} \otimes \overline{\mathcal{A}}^{\otimes(n+1)}$ is given simply by the shift

$$d(a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n) := 1 \otimes \bar{a}_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n. \quad (4.11)$$

Since $\bar{1} = 0$, we get $d^2 = 0$ at once. Evidently, starting from degree zero and applying d repeatedly gives

$$a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n = a_0\, da_1 \cdots da_n. \tag{4.12}$$

Next we make $\Omega^\bullet \mathcal{A}$ an $\mathcal{A}$-bimodule. The left module property is immediate:

$$a'(a_0\, da_1 \cdots da_n) = (a'a_0)\, da_1 \cdots da_n. \tag{4.13}$$

To get the right module property, one uses the postulated derivation property $da\, b = d(ab) - a\, db$ to pull the elements of $\mathcal{A}$ through to the left:

$$\begin{aligned}
&(a_0\, da_1 \cdots da_n) b_0 \\
&\quad = a_0\, da_1 \cdots da_{n-1}\, d(a_n b_0) - a_0\, da_1 \cdots da_{n-1}\, a_n\, db_0 = \cdots \\
&\quad = (-)^n a_0 a_1\, da_2 \cdots da_n\, db_0 \\
&\qquad + \sum_{i=1}^{n-1} (-)^{n-i} a_0\, da_1 \cdots d(a_i a_{i+1}) \cdots da_n\, db_0 \\
&\qquad + a_0\, da_1 \cdots da_{n-1}\, d(a_n b_0).
\end{aligned} \tag{4.14}$$

Lastly, we define $(a_0\, da_1 \cdots da_n)(b_0\, db_1 \cdots db_m) := ((a_0\, da_1 \cdots da_n) b_0)\, db_1 \times \cdots db_m$, so that $\Omega^\bullet \mathcal{A}$ becomes a graded algebra, as required.

Notice that $d(a_0\, da_1 \cdots da_n) = da_0\, da_1 \cdots da_n$ by (4.11) and (4.12). We also have the useful formula:

$$a_0\, [d, a_1] \cdots [d, a_n] 1 = a_0\, da_1 \cdots da_n, \tag{4.15}$$

where the a_j on the left hand side are regarded as left multiplication operators. This is easily verified by induction on n: note that $[d, a_n]1 = da_n - a_n\, d1 = da_n$, and $[d, a_{n-1}]\, da_n = d(a_{n-1}\, da_n) - a_{n-1}\, d(da_n) = da_{n-1}\, da_n$.

4.5. If $\mathcal{A}$ is an involutive algebra, $\Omega \mathcal{A}$ is made an involutive algebra by

$$(a_0\, da_1 \cdots da_n)^* := da_n^* \cdots da_1^*\, a_0^*. \tag{4.16}$$

One checks, using (4.14), that $(a_0\, da_1 \cdots da_n)^{**}$ telescopes to $a_0\, da_1 \cdots da_n$. From (4.14), it is also easy to check that $(a_0\, da_1 \cdots da_n\, b_0)^* = b_0^*\, da_n^* \cdots da_1^*\, a_0^*$. Thus we have $\omega^{**} = \omega$ and $(\omega\eta)^* = \eta^* \omega^*$ for $\omega, \eta \in \Omega^\bullet \mathcal{A}$, as required. Note that if $\alpha \in \Omega^1 \mathcal{A}$, then $(d\alpha)^* = -d\alpha^*$.

4.6. For later use, we give here a few simple identities for the differentials of idempotents. Suppose that $p^2 = p$ in $\mathcal{A}^{n \times n}$. We have $dp = d(p^2) = p\, dp + dp\, p$ in $\Omega^1 \mathcal{A}^{n \times n}$; hence $p\, dp = p^2\, dp + p\, dp\, p$, so that $p\, dp\, p = 0$. Thus also $p\, dp\, dp = p\, dp(p\, dp + dp\, p) = p\, dp\, dp\, p = dp\, dp\, p$ in $\Omega^2 \mathcal{A}^{n \times n}$. In summary:

$$p\, dp\, p = 0, \qquad p\, dp\, dp = dp\, dp\, p, \qquad dp\, p = (1 - p)\, dp. \tag{4.17}$$

4.7. We examine what the general construction yields for our two main examples, the algebra $C^\infty(M)$ and the two-point space.

Example 1. For $\mathcal{A} = C^\infty(M)$, we identify $\mathcal{A}^{\otimes n} = C^\infty(M \times \cdots \times M)$. If $f \in \mathcal{A}$, then $df(x_1,x_2) = (1 \otimes f - f \otimes 1)(x_1,x_2) = f(x_2) - f(x_1)$. Thus $\Omega^1\mathcal{A}$ is identified with the set of functions of two variables vanishing on the diagonal. More generally, $\Omega^n\mathcal{A}$ is identified with the set of functions of $n+1$ variables vanishing on contiguous diagonals. The differential is given by:

$$df(x_1,\dots,x_{n+1}) := \sum_{k=1}^{n+1} (-)^{k-1} f(x_1,\dots,x_{k-1},x_{k+1},\dots,x_{n+1}). \tag{4.18}$$

The left and right actions of $\mathcal{A}$ on $\Omega^n\mathcal{A}$ are given by:

$$\begin{aligned} (gf)(x_1,\dots,x_{n+1}) &:= g(x_1)f(x_1,\dots,x_{n+1}), \\ (fg)(x_1,\dots,x_{n+1}) &:= f(x_1,\dots,x_{n+1})g(x_{n+1}). \end{aligned} \tag{4.19}$$

The product of an m-form f and an n-form h is:

$$fh(x_1,\dots,x_{m+n+1}) := f(x_1,\dots,x_{m+1})\,h(x_{m+1},\dots,x_{m+n+1}). \tag{4.20}$$

Finally, the involution is:

$$f^*(x_1,\dots,x_{n+1}) := f(x_{n+1},\dots,x_1)^*. \tag{4.21}$$

Example 2. Take $\mathcal{A} = \mathbb{C}^2 \equiv \mathbb{C} \oplus \mathbb{C}$. Note first that $\Omega^\bullet\mathbb{C}$ is trivial. For $\mathcal{A} = \mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$, we have $\overline{\mathcal{A}} \simeq \mathbb{C}$, where we can identify the class of (w_1,w_2) with $w_1 - w_2$. It is clear that $\Omega^1(\mathbb{C}^2) = \mathbb{C}^2 \otimes \mathbb{C} = \mathbb{C}^2$; more generally, $\Omega^n(\mathbb{C}^2) = \mathbb{C}^2$ for any n. The differential on $\Omega^0\mathbb{C}^2$ and $\Omega^1\mathbb{C}^2$ is given by:

$$\begin{aligned} d(a_1,a_2) &= (a_2 - a_1, a_1 - a_2) \quad \text{for } a = (a_1,a_2) \in \Omega^0\mathbb{C}^2, \\ d(b_{12},b_{21}) &= (b_{21} + b_{12}, b_{12} + b_{21}) \quad \text{for } b = (b_{12},b_{21}) \in \Omega^1\mathbb{C}^2. \end{aligned} \tag{4.22}$$

Here we are adopting a convenient mnemonic device, based on the definition of $\Omega^1\mathcal{A}$ as a subspace of $\mathcal{A} \otimes \mathcal{A}$. We regard the elements of each $\Omega^k\mathbb{C}^2$ as indexed pairs: $a = (a_1,a_2) \in \Omega^0\mathbb{C}^2$, $b = (b_{12},b_{21}) \in \Omega^1\mathbb{C}^2$, $c = (c_{121},c_{212}) \in \Omega^2\mathbb{C}^2$, and so on, subject to the rule that adjacent indices should be unequal. This is indeed a particular case of the previous example, with $M = \{q_1,q_2\}$ just a two-point space.

It is convenient to introduce the "finite difference" $\Delta a := a_1 - a_2$, so that $da = \Delta a\,(-1,1)$ for $a \in \Omega^0\mathbb{C}^2$.

The left and right $\mathcal{A}$-actions on $\Omega^1\mathcal{A}$ are given by

$$\begin{aligned} (a_1,a_2)(b_{12},b_{21}) &= (a_1b_{12}, a_2b_{21}), \\ (b_{12},b_{21})(a_1,a_2) &= (b_{12}a_2, b_{21}a_1), \quad \text{for } a \in \mathcal{A},\ b \in \Omega^1\mathcal{A}. \end{aligned} \tag{4.23}$$

They are not the same, even though $\mathcal{A}$ is commutative. The last equality is checked by taking $b = a'\,da''$, so $ba = a'\,da''\,a = a'\,d(a''a) - a'a''\,da =$

$(-a_1' \Delta a'' a_2, a_2' \Delta a'' a_1) = (b_{12}a_2, b_{21}a_1)$, as claimed. A similar computation gives $(b_{12}, b_{21})^* = (-b_{21}^*, -b_{12}^*)$ in $\Omega^1(\mathbb{C}^2)$.

It is easily seen that $a\,da'\,da'' = (a_1\Delta a'\Delta a'', a_2\Delta a'\Delta a'')$ in $\Omega^2(\mathcal{A})$; a general element of $\Omega^2(\mathcal{A})$ is a sum of these. The left and right $\mathcal{A}$-actions on $\Omega^2\mathcal{A}$ are thus given by

$$\begin{aligned}(a_1, a_2)(c_{121}, c_{212}) &= (a_1c_{121}, a_2c_{212}),\\ (c_{121}, c_{212})(a_1, a_2) &= (c_{121}a_1, c_{212}a_2), \quad \text{for } a \in \mathcal{A},\ c \in \Omega^2\mathcal{A}.\end{aligned} \tag{4.24}$$

Finally, $(c_{121}, c_{212})^* = (c_{121}^*, c_{212}^*)$ for $c \in \Omega^2(\mathcal{A})$.

With the index notation we are using here (and which will eventually generalize to the differential algebra underlying the Standard Model), one sees that all the algebraic laws follow by matching adjacent indices: for example, $(ab)_{ij} = a_ib_{ij}$, $(ba)_{ij} = b_{ij}a_j$, $(da)_{ij} = a_j - a_i$, $(b^*)_{ij} := -b_{ji}^*$, $(ac)_{ijk} = a_ic_{ijk}$, $(ca)_{ijk} = c_{ijk}a_k$, and $(c^*)_{ijk} = c_{kji}^*$. As exercises we leave the identities $(bb')_{ijk} = b_{ij}b'_{jk}$ and $(db)_{ijk} = b_{jk} - b_{ik} + b_{ij}$.

We introduce another useful notational convention. Let $p = (1,0)$ a self-adjoint idempotent in $\mathcal{A}$. Then $dp = (-1,1)$. In $\Omega^1\mathcal{A}$ one has the basis $\{p\,dp, (1-p)\,dp\} = \{(-1,0),(0,1)\}$. In particular, we may rewrite the derivation rule for $a \in \Omega^0\mathbb{C}^2$ as $da = \Delta a(p\,dp + (1-p)\,dp)$.

4.8. The reader will probably feel entitled to know what the relation is, in the commutative case $\mathcal{A} = C^\infty(M)$, between $\Omega^\bullet\mathcal{A}$ and the usual algebra of differential forms $\mathcal{E}^\bullet\mathcal{A} = \Gamma(\Lambda^\bullet T^*M)$, besides the fact that there is a covering homomorphism from the first to the second (the kernel of this "universal" map is rather large). Roughly, the answer is as follows: a Hochschild type complex can be defined on $\Omega^\bullet\mathcal{A}$ such that the corresponding homology *classes* can be identified with the differential forms. This is done in the spirit of cyclic cohomology, which we will now briefly examine.

We return to the chain complex $C_\bullet(\mathcal{A}) = \mathcal{A}^{\otimes\bullet+1}$ but we replace the boundary operator b' of (4.2) by:

$$\begin{aligned}b(a_0 \otimes a_1 \otimes \cdots \otimes a_n) :=\ & b'(a_0 \otimes a_1 \otimes \cdots \otimes a_n)\\ &+ (-1)^n a_na_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}.\end{aligned} \tag{4.25}$$

For example, $b(a_0 \otimes a_1) = a_0a_1 - a_1a_0$, $b(a_0 \otimes a_1 \otimes a_2) = a_0a_1 \otimes a_2 - a_0 \otimes a_1a_2 + a_2a_0 \otimes a_1$. One checks that $b^2 = 0$. The homology of this complex [4] is called the *Hochschild homology* $H_\bullet(\mathcal{A},\mathcal{A})$. [This notation is used because the last term in (4.25) and the first term in (4.2) involve the products a_na_0 and a_0a_1, so one can replace the lowest-degree copy of $\mathcal{A}$ by any $\mathcal{A}$-bimodule M, yielding a homology $H_\bullet(\mathcal{A}, M)$.]

The chains of the form $a = a_0 \otimes \cdots \otimes 1 \otimes \cdots \otimes a_n$ with $a_k = 1$ for some $k > 0$, generate a subcomplex $D_\bullet\mathcal{A}$ since $a \in D_n\mathcal{A}$ entails $ba \in D_{n-1}\mathcal{A}$. From (4.3) and (4.25) one obtains $sb(a) + bs(a) = a$ for $a \in D_n\mathcal{A}$ with $a_n = 1$; by composing s with a cyclic permutation σ of the factors so that $\sigma(a)_n = 1$, we obtain a chain homotopy s' satisfying $s'b(a) + bs'(a) = a$ for $a \in D_n\mathcal{A}$, $n > 0$. Therefore this subcomplex is acyclic, i.e., $H_n(D_\bullet\mathcal{A}, b) = 0$ for $n > 0$. The quotient bimodule $C_n\mathcal{A}/D_n\mathcal{A}$ can be identified with $\mathcal{A} \otimes \overline{\mathcal{A}}^{\otimes n} = \Omega^n(\mathcal{A})$ for $n > 0$. So we can think of (4.25) as defining a boundary operator on the graded algebra $\Omega^\bullet\mathcal{A}$. We can therefore rewrite it as

$$\begin{aligned} b(a_0\, da_1 \cdots da_n) := {} & a_0 a_1\, da_2 \cdots da_n \\ & + \sum_{i=1}^{n-1} (-)^i a_0\, da_1 \cdots d(a_i a_{i+1}) \cdots da_n \\ & + (-1)^n a_n a_0\, da_1 \cdots da_{n-1}. \end{aligned} \tag{4.26}$$

A comparison with (4.14) makes it clear that this formula may be rewritten more compactly as $b(\omega\, da) = (-)^n[\omega, a]$ for $\omega \in \Omega^n(\mathcal{A})$.

4.9. Several important quantities in noncommutative geometry are Hochschild *cocycles*. By definition, an n-cochain is an $(n+1)$-linear functional on $\mathcal{A}$. This is the same thing as a linear form on $\mathcal{A}^{\otimes(n+1)}$, or an n-linear form on $\mathcal{A}$ with values in the dual space $\mathcal{A}'$. We mention that $\mathcal{A}'$ is an $\mathcal{A}$-bimodule, where we put $(a\psi b)(c) := \psi(bca)$ for $\phi \in \mathcal{A}'$. The coboundary operator, also called b, is given by transposing (4.25):

$$\begin{aligned} b\varphi(a_0, \dots, a_{n+1}) := {} & \sum_{i=0}^{n} (-)^i \varphi(a_0, \dots, a_i a_{i+1}, \dots, a_{n+1}) \\ & + (-1)^{n+1}\varphi(a_{n+1}a_0, \dots, a_n). \end{aligned} \tag{4.27}$$

The cohomology of this complex is the Hochschild cohomology, conventionally denoted by $H^\bullet(\mathcal{A}, \mathcal{A}')$ (the first and last terms of (4.27) use the bimodule property of $\mathcal{A}'$).

Suppose that φ is a Hochschild cochain such that $\varphi(a_0, \dots, a_n) = 0$ whenever $a_k = 1$ for some $k = 1, 2, \dots, n$; we may call φ a *reduced* Hochschild cochain [4]. Then there is a naturally associated linear map $\hat\varphi: \Omega^n\mathcal{A} \to \mathbb{C}$ given by

$$\hat\varphi(a_0\, da_1 \cdots da_n) := \varphi(a_0, \dots, a_n). \tag{4.28}$$

It is clear from (4.27) that the reduced cochains form a subcomplex of the Hochschild complex (since $b\varphi$ is reduced whenever φ is). Whether one uses the reduced complex or not is a matter of convenience, since the full Hochschild complex for $\mathcal{A}$ can be identified with the reduced complex for the augmented algebra $\mathcal{A}^+$ with the degrees of the chains shifted by one [14].

If we restrict to continuous multilinear functionals, and take $\mathcal{A} = C^\infty(M)$, then the antisymmetrization of a reduced cochain φ with respect to all arguments but the first yields a de Rham current C_φ:

$$\langle C_\varphi, a_0\, \boldsymbol{d}a_1 \wedge \cdots \wedge \boldsymbol{d}a_n \rangle := \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, \varphi(a_0, a_{\sigma(1)}, \ldots, a_{\sigma(n)}). \tag{4.29}$$

The notation $\boldsymbol{d}$ means simply the ordinary exterior derivation of forms; we write it thus to distinguish it from the universal d, which in the commutative case is the finite difference operator (4.18).

One checks that antisymmetrizing takes cocycles to cocycles and kills coboundaries. This yields the aforementioned map from Hochschild cohomology classes to de Rham currents. It can be verified [7] that this map is an isomorphism. An isomorphism theorem, which in a sense is a dual result to the above, is proved in ref. [24].

4.10. A Hochschild zero-cocycle τ is a trace: $\tau \in \mathcal{A}'$ and $b\tau(a_0, a_1) = \tau(a_0 a_1) - \tau(a_1 a_0) = 0$. To extend the trace property to higher orders, we say an n-cochain φ is *cyclic* if $\lambda\varphi = \varphi$, where

$$\lambda\varphi(a_0, \ldots, a_n) := (-)^n \varphi(a_n, a_0, \ldots, a_{n-1}). \tag{4.30}$$

[The $(-)^n$ is the sign of the cyclic permutation.] For a cyclic one-cocycle we then have

$$\varphi(a_0, a_1) = -\varphi(a_1, a_0), \quad \varphi(a_0 a_1, a_2) - \varphi(a_0, a_1 a_2) + \varphi(a_2 a_0, a_1) = 0. \tag{4.31}$$

Clearly, a cyclic one-coboundary is a linear function of the commutator:

$$\varphi(a_0, a_1) = b\psi(a_0, a_1) = \psi([a_0, a_1]). \tag{4.32}$$

If φ is a reduced cyclic cocycle, then from (4.14) and (4.28) one can check that

$$\hat\varphi(a_0\, da_1 \cdots da_n\, b_0\, db_1 \cdots db_m) = (-)^{nm} \hat\varphi(b_0\, db_1 \cdots db_m\, a_0\, da_1 \cdots da_n). \tag{4.33}$$

In other words, $\hat\varphi$ is a graded trace on the graded algebra $\Omega^\bullet \mathcal{A}$.

Let us also introduce the *cyclic antisymmetrizer* $N := \mathrm{id} + \lambda + \cdots + \lambda^n$. From (4.27) and (4.30) we find, after a brief struggle, that $bN = Nb'$, where b' denotes the transpose of (4.2). Clearly any $N\varphi$ is cyclic, since $\lambda N = N$. On the other hand, if $\lambda\varphi = \varphi$, then $N((n+1)^{-1}\varphi) = \varphi$, so that every cyclic cochain lies in the image of N.

We denote the space of cyclic n-cochains by $CC^n(\mathcal{A}) := \ker(\lambda - \mathrm{id}) = \operatorname{im} N$. The identity $bN = Nb'$ shows that b preserves $CC^n(\mathcal{A})$, so that $(CC^\bullet(\mathcal{A}), b)$ is a subcomplex of the Hochschild cochain complex.

A *cyclic cocycle* is a cyclic cochain satisfying $b\varphi = 0$. Every cyclic cocycle is in particular a Hochschild cocycle, but not every Hochschild coboundary $b\psi$ which is cyclic is a cyclic coboundary, since ψ might not itself be cyclic. Thus the *cyclic cohomology* of $\mathcal{A}$, which we will denote by $HC^\bullet(\mathcal{A})$, is different from the Hochschild cohomology. Both cohomologies play a rôle in the calculation of Yang–Mills functionals, as we will see.

4.11. **Definition.** Let $\mathcal{E}$ be a hermitian vector bundle over a $*$-algebra $\mathcal{A}$. A (universal) *connection* on $\mathcal{E}$ is a linear map $\nabla: \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A}$ such that

$$\nabla(sa) = (\nabla s)a + s \otimes da, \quad \text{for all } s \in \mathcal{E},\ a \in \mathcal{A}; \tag{4.34}$$

this connection is compatible with the hermitian structure if

$$d(s \mid s') = (\nabla s \mid s') + (s \mid \nabla s'), \quad \text{for all } s, s' \in \mathcal{E}. \tag{4.35}$$

Note that $\mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A}$ is a right $\mathcal{A}$-module.

Compatible connections always exist. If $\mathcal{E} \simeq \mathcal{A}^m$ is free, then $\mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A} \simeq (\Omega^1\mathcal{A})^m$, and d extended componentwise to $\mathcal{A}^m$ defines a compatible connection on $\mathcal{E}$.

Now, consider $\mathcal{E} \simeq p\mathcal{A}^m$, where p is a self-adjoint idempotent in $\mathcal{A}^{m\times m}$. Regarding $\mathcal{E}$ as a submodule of $\mathcal{A}^m$, we see that pd defines a compatible connection on $\mathcal{E}$: if $s = (s_1, \dots, s_m) = (s_i)$, $ps = (p_i^j s_j)$, then $pd(sa) = (p_i^j\, d(s_j a)) = (p_i^j (ds_j)a + p_i^j s_j\, da) = (p\, ds)a + s\, da$ whenever $ps = s$. Moreover, $d(s \mid t) = d(\sum_i s_i^* t_i) = \sum_i ds_i^*\, t_i + s_i^*\, dt_i = (ds \mid pt) + (ps \mid dt) = (p\, ds \mid t) + (s \mid p\, dt)$.

If ∇_1, ∇_2 are two connections on $\mathcal{E}$, then $\nabla_1 - \nabla_2 =: \alpha \in \mathrm{End}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A})$. If both connections are compatible, then α is skew-adjoint. Thus any compatible connection is of the form $\nabla = pd + \alpha$ with $\alpha^* = -\alpha$. If we identify $\mathcal{E} \simeq p\mathcal{A}^m$, we can regard α as an element of $\mathcal{A}^{m\times m} \otimes_{\mathcal{A}} \Omega^1\mathcal{A}$ such that $\alpha = \alpha p = p\alpha = p\alpha p$. Conversely, given such a universal one-form α, the map $\nabla := pd + \alpha$ is indeed a compatible connection:

$$\nabla(sa) := p\, d(sa) + \alpha sa = p\, dsa + \alpha sa + ps\, da = (\nabla s)a + s\, da, \tag{4.36}$$

where again $ps = s$ is used.

4.12. We now extend ∇ to a derivation of $\mathcal{E}$-valued forms, $\nabla: \mathcal{E} \otimes_{\mathcal{A}} \Omega^\bullet\mathcal{A} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^{\bullet+1}\mathcal{A}$, by requiring

$$\nabla(s \otimes \omega) = (\nabla s) \otimes \omega + s \otimes d\omega, \quad \forall s \in \mathcal{E},\ \omega \in \Omega^\bullet\mathcal{A}. \tag{4.37}$$

Clearly, this extension is unique. Moreover, if ω is homogeneous, we get from (4.9)

$$\nabla((s \otimes \omega)\eta) = \nabla(s \otimes \omega\eta) = \nabla(s \otimes \omega)\eta + (-)^{\deg \omega}(s \otimes \omega)\, d\eta. \tag{4.38}$$

We can now also define the square $\theta^\nabla := \nabla^2$, usually abbreviated θ, and call it the *curvature* of ∇.

We compute the general form for the curvature:

$$\begin{aligned}\theta(s) &= p\,d(p\,ds + \alpha s) + \alpha(p\,ds + \alpha s) \\ &= p\,dp\,ds + p\,d\alpha\,s - p\alpha\,ds + \alpha p\,ds + \alpha^2 s \\ &= (p\,dp\,dp + p\,d\alpha\,p + \alpha^2)s, \end{aligned} \tag{4.39}$$

where we have used (4.17). The result of the computation shows that, in contradistinction to ∇, the curvature is an $\mathcal{A}$-linear operator. One may then regard θ as an element of $\mathrm{End}_{\mathcal{A}}(\mathcal{E}) \otimes_{\mathcal{A}} \Omega^2\mathcal{A}$, which is to say, as a matrix of universal two-forms satisfying $\theta = p\theta p$.

There is a natural action of the group of gauge transformations $\mathcal{U}(\mathcal{E})$ on the space of compatible connections, given by

$$\gamma_u(\nabla) := u\nabla u^* : s \mapsto u\nabla(u^* s), \tag{4.40}$$

with curvature $u\theta u^*$. If $\mathcal{E} = p\mathcal{A}^m$ and $\nabla = pd + \alpha$, then $\gamma_u(\nabla)s = u(pd + \alpha)u^*s = pu\,d(u^*s) + u\alpha u^*s = (pd + \gamma_u(\alpha))s$, where we define the action of $\mathcal{U}(\mathcal{E})$ on matrices of one-forms to be

$$\gamma_u(\alpha) := u\,du^* + u\alpha u^*. \tag{4.41}$$

Example 1. If $\mathcal{E} = \mathcal{A}$, call $\alpha = \nabla 1$; then clearly $\nabla f = df + \alpha f$ and $\theta = d\alpha + \alpha^2$. Even in this almost trivial commutative case $\mathcal{A} = C^\infty(M)$, we have a difference with respect to the classical expression, as now $\alpha^2 \neq 0$.

Example 2. When $\mathcal{E} = \mathcal{A} = \mathbb{C}^2$, the two-point space, a compatible connection is specified by an element of $\Omega^1\mathbb{C}^2$ of the form $\alpha = (\phi^* - 1, \phi - 1) = (1 - \phi^*)p\,dp + (\phi - 1)(1 - p)\,dp$. (The reason we write $\phi - 1$ rather than $\phi \in \mathbb{C}$ will become clear later.) We have for the curvature, using (4.17):

$$d\alpha + \alpha^2 = -\{(\phi - 1) + (\phi - 1)^* + |\phi - 1|^2\}\,dp\,dp = (1 - |\phi|^2)\,dp\,dp. \tag{4.42}$$

The gauge group $\mathrm{U}(1) \times \mathrm{U}(1)$ action on connections is found from (4.41). Since $u\,du^* = (1 - u_1u_2^*)p\,dp + (u_1^*u_2 - 1)(1 - p)\,dp$, we obtain $\gamma_u(\alpha) = (1 - u_1u_2^*\phi^*)p\,dp + (u_1^*u_2\phi - 1)(1 - p)\,dp$. Thus the gauge action is just multiplication of ϕ by $u_1^*u_2$.

5. Noncommutative geometry: K-cycles

5.1. In this section we pull the previous strands together. We introduce the basic object of noncommutative integrodifferential calculus, the K-cycle.

Definition. A *K-cycle* $(\mathcal{H}, D)$ on the $*$-algebra $\mathcal{A}$ consists of a unitary representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, together with an (unbounded) self-adjoint operator D on $\mathcal{H}$ with compact resolvent, such that $[D, a]$ is bounded for all

$a \in \mathcal{A}$. In many cases, $\mathcal{H}$ is a $\mathbb{Z}_2$-graded Hilbert space, equipped with a grading operator Γ such that $\Gamma^2 = 1$, $\mathcal{A}$ acts on $\mathcal{H}$ by even operators, and D is an odd operator, i.e., $a\Gamma = \Gamma a$ for $a \in \mathcal{A}$, and $D\Gamma = -\Gamma D$.

To show the usefulness of this definition, we take up the important commutative example of the Dirac operator on a compact spin^c manifold. The Dirac operator is important for several reasons, as we shall see. Firstly, as a distinguished square root of the Laplacian, in some sense (to be made clearer below) it incorporates the entire Riemannian geometry of the underlying manifold. Secondly, as a first-order elliptic pseudodifferential operator, it provides a bridge to index theory and thus to manifold invariants, such as the Chern character, so it is of manifest importance for gauge-invariant phenomena. Thirdly, it determines a special cyclic cocycle, which indeed gives the dual object to the Chern character [7]. If a Dirac operator is available, classical gauge theory can be lifted to the purely operator level, ripe for generalization.

We take $\mathcal{A} = C^\infty(M)$. Let $\mathcal{H} := L^2(S)$, the space of square integrable sections of the irreducible spinor bundle S over M, and D the corresponding Dirac operator (for precise definitions, see the appendix). Recall that $\mathcal{A}$ acts on $\mathcal{H}$ by multiplication operators, i.e., multiplication by scalars on each fibre of S.

Lemma 5.1. *The densely defined operator $[D, a]$ is bounded.*

Proof. The operator $[D, a]$ should be viewed as a quadratic form, its boundedness meaning that $|\langle Ds' \mid as\rangle - \langle a^* s' \mid Ds\rangle| \leq C\|s\|\,\|s'\|$ for all sections s, s' in the domain of D. We introduce the Lipschitz norm $\|a\|_{\mathrm{Lip}} := \sup_{p\neq q} |a(p) - a(q)|/d(p,q)$ with respect to the geodesic distance

$$d(p,q) := \inf_\gamma l_\gamma(p,q), \tag{5.1}$$

where $l_\gamma(p,q)$ is the length of the path γ from p to q.

If $a \in C^\infty(M)$, the Lipschitz norm of a is the essential supremum $\|\boldsymbol{d}a\|_\infty$; recall that $\boldsymbol{d}a \in \mathcal{E}^1(M) = \Gamma(T^*M)$ denotes the ordinary differential of a. We remark that $D(as) = a\,Ds + c(\boldsymbol{d}a)s$, as is clear from proposition A.7. Thus $[D, a]$ is nothing but Clifford multiplication by $\boldsymbol{d}a$, i.e., the action of $c(\boldsymbol{d}a)$ on $L^2(S)$. The operator norm of $c(\boldsymbol{d}a)$ equals $\|\boldsymbol{d}a\|_\infty$. Hence:

$$\|[D,a]\| = \|\boldsymbol{d}a\|_\infty = \|a\|_{\mathrm{Lip}}. \tag{5.2}$$

See also ref. [3, proposition 3.38]. □

We note that the algebra of Lipschitz functions is uniformly dense in $C(M)$.

The metric on M may also be recovered from the Dirac operator, as shown by the following lemma [9].

Lemma 5.2. *The geodesic distance between two points p, q of M is given by*

$$d(p,q) = \sup\{\, |a(p) - a(q)| : a \in \mathcal{A},\ \|[D,a]\| \le 1 \,\}. \tag{5.3}$$

Proof. It is clear that $d(p,q)$ is majorized by the right hand side of (5.3), by taking $a(q) := d(p,q)$; this is a Lipschitz function with constant 1, so that $\|[D,a]\| \le 1$. The converse inequality follows from (5.2). □

The formula (5.3) for the distance is dual to the original formula (5.1), in that, instead of involving functions from $\mathbb{R}$ to M, it involves functions from M to $\mathbb{R}$. In the case of discrete or noncommutative spaces there is a scarcity of arcs, but there are plenty of "functions", i.e., the elements of $\mathcal{A}$ itself. The right hand side of (5.3) defines also a distance on the space of states of a smooth algebra (equipped with a K-cycle), so it admits a natural noncommutative generalization. Note finally that (5.1) is suspect from the point of view of quantum mechanical operationalism.

5.2. Another important element of Riemannian geometry is the canonical volume measure:

$$\mu(dx) = \sqrt{\det g}\, \boldsymbol{d}x^1 \wedge \cdots \wedge \boldsymbol{d}x^n, \quad n = \dim M. \tag{5.4}$$

The elliptic pseudodifferential operator D has a finite-dimensional kernel. To avoid the irrelevant complications which arise if this kernel is not zero (which can be dealt with, for instance, by adding a "mass term" [7]), we will adopt from now on the notation that $\ker D = \{0\}$, so that D has a bounded inverse.

We now recover the measure from the Dirac K-cycle in the following way.

Theorem 5.3. *For $a \in C^\infty(M)$:*

$$C_n \int_M a(x)\, \mu(dx) = \mathrm{Tr}^+ (a|D|^{-n}), \tag{5.5}$$

where the constant C_n is given by

$$C_{2k} = \frac{(2\pi)^{-k}}{k!} \quad \text{or} \quad C_{2k+1} = \frac{\pi^{-k-1}}{(2k+1)!!}.$$

Proof. This is an easy consequence of the trace theorem. Since $a \in \mathcal{A}$ is a bounded multiplication operator, $a|D|^{-n}$ is a pseudodifferential operator of order $-n$. Its principal symbol is just $p_{-n}(x,\xi) = a(x)\|\xi\|^{-n}$, which reduces to the scalar matrix $a(x)$ (of rank $2^{\lfloor n/2 \rfloor} = \dim S_x$) on the cosphere bundle $\|\xi\| = 1$. Thus $\mathrm{Tr}^+(a|D|^{-n}) = C_n \int_M a\,\mu$, with $C_n = 2^{\lfloor n/2 \rfloor}\Omega_n / n(2\pi)^n$, which simplifies to the stated values. □

Definition. A K-cycle is said to be n^+ *summable* if $|D|^{-1} \in \mathcal{L}^{n+}(\mathcal{H})$.

Equivalently, a K-cycle is n^+ summable if $|D|^{-n}$ belongs to the Dixmier ideal. This is the case in the commutative example we have been examining, with $n = \dim M$, and so the order of summability may be regarded as the dimension of the K-cycle $(\mathcal{H}, D)$. Indeed, from theorem 2.3 we have $(1 + D^2)^{-1/2} \in \mathcal{L}^{n+}(\mathcal{H})$, since $D^2 = -\Delta$, and $|D|^{-1} = T(1 + D^2)^{-1/2}$ with T bounded.

5.3. The availability of a K-cycle allows us to project from the algebra of universal forms $\Omega^\bullet\mathcal{A}$ to a more useful graded differential algebra. It will transpire that, in the Riemannian case, we need not to go through the Hochschild complex, in order to descend to the de Rham algebra.

Proposition 5.4. *Given any K-cycle $(\mathcal{H}, D)$ over an algebra $\mathcal{A}$, the following equality defines a $*$-representation of $\Omega^\bullet\mathcal{A}$ in $\mathcal{H}$:*

$$\pi(a_0\, da_1 \cdots da_n) := \mathrm{i}^n a_0\, [D, a_1] \cdots [D, a_n]. \tag{5.6}$$

Proof. That π is a homomorphism is evident, since both d and $[D, \cdot]$ are derivations on $\mathcal{A}$. Also, since $[D, a]^* = -[D, a^*]$ by self-adjointness of D, we get $\pi(a_0\, da_1 \cdots da_n)^* = \pi((a_0\, da_1 \cdots da_n)^*)$, using (4.16). □

However, the π homomorphism is not necessarily a *differential* one. If $\sum_j a_0^j [D, a_1^j]$ is a denizen of $\pi(\Omega^1\mathcal{A})$, we would like to define $d \sum_j a_0^j [D, a_1^j]$ as $\mathrm{i} \sum_j [D, a_0^j][D, a_1^j]$; however, the same $\sum_j a_0^j [D, a_1^j]$ could be represented in several ways using elements of $\mathcal{A}$, so the expression $\sum_j [D, a_0^j][D, a_1^j]$ is potentially ambiguous. Indeed, $\pi b = 0$ for $b \in \Omega^k\mathcal{A}$ does not in general imply $\pi(db) = 0$, so the ambiguity does occur.

We prove this by a simple but important example. Let $b = a\, da - \frac{1}{2} d(a^2) \in \Omega^1\mathcal{A}$. In the Riemannian commutative case, πb is Clifford multiplication by $\mathrm{i}\, a\, \boldsymbol{d}a - \frac{1}{2}\mathrm{i}\boldsymbol{d}(a^2)$; thus $\pi b = 0$. However, $\pi(db) = -c(\boldsymbol{d}a)c(\boldsymbol{d}a) = q(\boldsymbol{d}a, \boldsymbol{d}a) \neq 0$.

We need the following technical lemma.

Lemma 5.5. *Let J_0 be the graded two-sided ideal given by $J_0^k := \{b \in \Omega^k\mathcal{A} : \pi b = 0\}$. Then $J := J_0 + dJ_0$ is a graded differential two-sided ideal of $\Omega^\bullet\mathcal{A}$.*

Proof. We must show that $J_0 + dJ_0$ is still an ideal. Let $b \in J^k$ be a homogeneous element of J. Write it as $b = b_1 + db_2$, where $b_1 \in J_0^k$, $b_2 \in J_0^{k-1}$. For $c \in \Omega^l\mathcal{A}$, we obtain $bc = b_1 c - (-)^k b_2\, dc + d(b_2 c)$. Since $b_1 c - (-)^k b_2\, dc \in J_0$ and $b_2 c \in J_0$ also, we get $bc \in J$. The computation for cb proceeds similarly. Finally, J is a differential ideal, as clearly $dJ = dJ_0 \subset J$. □

Definition. Let $(\mathcal{H}, D)$ be a K-cycle over an algebra $\mathcal{A}$. We define the graded differential algebra of *D-forms* on $\mathcal{A}$ as

$$\Omega_D^\bullet \mathcal{A} := \pi(\Omega^\bullet \mathcal{A})/J. \tag{5.7}$$

The canonical projection from $\Omega^\bullet(\mathcal{A})$ to $\Omega_D^\bullet(\mathcal{A})$ will be written π_D.

In order to grasp the meaning of (5.7), we consider $\Omega_D^k \mathcal{A}$, for $k = 0, 1, 2$.

– Clearly $J_0^0 = J^0 = \{0\}$, so $\Omega_D^0 \mathcal{A} = \mathcal{A}$ as expected.

– Next, $J^1 = J_0^1 + d J_0^0 = J_0^1$, thus $\Omega_D^1 \mathcal{A}$ is the quotient of Ω^1 by $\ker \pi$. When $\mathcal{A} = C^\infty(M)$ and D is a Dirac operator, its elements operate by Clifford multiplication $c(\omega)$ on spinors, where ω is a one-form $\mathrm{i} \sum_j a_0^j \boldsymbol{d} a_1^j$.

– Finally, $J^2 = J_0^2 + d J_0^1$. Thus $\Omega_D^2 \mathcal{A} \simeq \pi(\Omega^2 \mathcal{A})/\pi(d J_0^1)$. More concretely, notice that the vector space of operators of the form

$$\{ \textstyle\sum_j [D, a_0^j][D, a_1^j] : a_0^j, a_1^j \in \mathcal{A},\ \sum_j a_0^j [D, a_1^j] = 0 \}, \tag{5.8}$$

form a subbimodule of $\pi(\Omega^2 \mathcal{A})$, since the side condition entails the identities

$$a'(\textstyle\sum_j [D, a_0^j][D, a_1^j]) = \sum_j [D, a' a_0^j][D, a_1^j],$$
$$(\textstyle\sum_j [D, a_0^j][D, a_1^j]) a' = \sum_j [D, a_0^j][D, a_1^j a'] - \sum_j [D, a_0^j a_1^j][D, a'], \tag{5.9}$$

whose right hand sides also belong to (5.8); in the second case, this follows simply from $\sum_j (a_0^j [D, a_1^j a'] - a_0^j a_1^j [D, a']) = (\sum_j a_0^j [D, a_1^j]) a' = 0$. Therefore, an element of $\Omega_D^2 \mathcal{A}$ is a class of elements of the form:

$$\textstyle\sum_j a_0^j [D, a_1^j][D, a_2^j], \quad a_0^j, a_1^j, a_2^j \in \mathcal{A}, \tag{5.10}$$

modulo the subbimodule (5.8).

The ambiguity in the definition of db for $b \in \pi(\Omega^1 \mathcal{A})$ no longer arises in the context of $\Omega_D^\bullet \mathcal{A}$. More generally, we have

$$\Omega_D^k \mathcal{A} \simeq \pi(\Omega^k)/\pi(d J_0^{k-1}). \tag{5.11}$$

5.4. We now specialize again to the commutative Riemannian case. Here, for $a_0, \dots, a_k \in \mathcal{A}$, one has $\pi(a_0\, da_1 \cdots da_k) = \mathrm{i}^n a_0\, c(\boldsymbol{d} a_1 \cdots \boldsymbol{d} a_k)$, where the differentials $\boldsymbol{d} a_j$ and their products are considered as sections of the Clifford algebra bundle $\mathbb{C}\mathrm{l}(M)$.

For each $x \in M$, and $k = 0, 1, \dots, n$, let C_x^k denote the subspace of the Clifford algebra fibre $\mathbb{C}\mathrm{l}_x$ of $\mathbb{C}\mathrm{l}(M)$ generated by products of at most k cotangent vectors in $T_x^* M$. Using the canonical inner product on $\mathbb{C}\mathrm{l}_x$ given by the trace of the spin representation, we can identify $\Lambda_{\mathbb{C}}^k T_x^* M$ with the orthogonal complement $C_x^k \ominus C_x^{k-1}$. Let us denote the quotient map from C_x^k to $\Lambda_{\mathbb{C}}^k T_x^* M$ by σ_k; we have

$$\sigma_k(v_1 \cdots v_k) = v_1 \wedge \cdots \wedge v_k, \quad \text{for } v_1, \dots, v_k \in T_x^* M. \tag{5.12}$$

Lemma 5.6. *Let $(\mathcal{H}, D)$ be the Dirac K-cycle on the algebra $C^\infty(M)$, and let $k \in \mathbb{N}$. A pair of operators (A_1, A_2) on $\mathcal{H}$ is of the form $A_1 = \pi(b)$, $A_2 = \pi(db)$*

for some $b \in \Omega^k \mathcal{A}$ iff there exist sections s_1, s_2 of C^k and C^{k+1}, respectively, such that

$$A_1 x = c(s_1)x, \qquad A_2 x = c(s_2)x, \tag{5.13a}$$

$$\mathrm{i}\,\boldsymbol{d}\sigma_k(s_1) = \sigma_{k+1}(s_2). \tag{5.13b}$$

Proof. If $b = a_0\,da_1 \cdots da_k$, it is clear that $\pi(b) = \mathrm{i}^k a_0\,\boldsymbol{d}a_1 \cdots \boldsymbol{d}a_k \in \Gamma(C^k)$ and $\pi(db) = \mathrm{i}^{k+1}\boldsymbol{d}a_0\,\boldsymbol{d}a_1 \cdots \boldsymbol{d}a_k \in \Gamma(C^{k+1})$. In this case

$$\mathrm{i}\,\boldsymbol{d}\sigma_k(s_1) = \mathrm{i}^{k+1}\boldsymbol{d}a_0 \wedge \cdots \wedge \boldsymbol{d}a_k = \sigma_{k+1}(s_2)\,.$$

Conversely, if $s_1 \in \Gamma(C^k)$ and $s_2 \in \Gamma(C^{k+1})$ satisfy $\mathrm{i}\,\boldsymbol{d}\sigma_k(s_1) = \sigma_{k+1}(s_2)$, then s_2 is determined by s_1 up to an ambiguity in $\Gamma(C^k)$. We may therefore suppose that $s_1 = 0$ and $s_2 \in \Gamma(C^{k-1})$. If we set $b' := (a_0\,da_0 - \frac{1}{2}d(a_0^2))\,da_1 \cdots da_{k-1} \in \Omega^k\mathcal{A}$, we have $\pi(b') = 0$, $\pi(db') = \|\boldsymbol{d}a_0\|^2\,\boldsymbol{d}a_1 \cdots \boldsymbol{d}a_{k-1}$. Since terms of the latter type generate $\Gamma(C^{k-1})$ as a $C^\infty(M)$-module, we can find $b \in \Omega^k\mathcal{A}$ with $\pi(b) = 0$ and $\pi(db)$ equal to any given element of $\Gamma(C^{k-1})$. □

[We can give an example of the above relations in physicists' language. In the trivial bundle $\mathcal{E} = \mathcal{A}$, a connection is defined by a universal one-form $\alpha = \sum a_j\,db_j$. Suppose $\pi_D(\alpha) = A = A_\mu\,dx^\mu$. We will have then $\pi(\alpha) = \gamma^\mu A_\mu$. We have $\pi(\alpha^2) = A^2$ clearly, whereas $\pi_D(\alpha)^2 = A \wedge A = 0$. Moreover, $\pi(d\alpha)$ is certainly not $\boldsymbol{d}A$. Writing $d\alpha = \sum da_j\,db_j$ and performing the necessary calculations, we get $\pi(d\alpha) = \frac{1}{2}F_{\mu\nu}\gamma^\mu\gamma^\nu + \phi + \partial \cdot A$ where the scalar field $\phi = -g^{\mu\nu}\sum a_j \partial_\mu \partial_\nu b_j$ and $\partial \cdot A := \partial^\nu A_\nu$. We thus have $\pi(\theta) = \pi(d\alpha + \alpha^2)$, but only the first term of $\pi(d\alpha)$ can be identified with the "Maxwell form" $\boldsymbol{d}A$.]

We finally recognize that $\Omega_D^\bullet C^\infty(M)$ is isomorphic to the de Rham algebra $\mathcal{E}^\bullet(M)$ of everyday forms and that the relation between $\pi(\Omega^\bullet\mathcal{A})$ and $\Omega_D^\bullet\mathcal{A}$ in this example is just the relation between the space of sections of the Clifford algebra—we can identify $\pi(\Omega^k\mathcal{A})$ and $\Gamma(C^k)$—and the space of sections of $\Lambda_{\mathbb{C}}^k T^*M$. Indeed, the above proof shows that if $s \in \Gamma(C^k)$ with $\sigma_k(s) = 0$, then $s = \pi(db)$ for some b with $\pi b = 0$; in other words,

$$\pi(dJ_0^{k-1}) = c(\ker \sigma_k). \tag{5.14}$$

It is also clear from (5.12) that if $t \in \Gamma(C^l)$, then $\sigma_{k+l}(st) = \sigma_k(s) \wedge \sigma_l(t)$ in $\mathcal{E}^{k+l}(M)$. Thus the symbol maps σ_k combine to yield an *isomorphism of graded algebras* $\sigma_D\colon \Omega_D^\bullet C^\infty(M) \to \mathcal{E}^\bullet(M)$. This is also an isomorphism of $C^\infty(M)$-modules. We note that we cannot at this stage say that σ_D is isometric, since no hermitian structure has yet been given on $\Omega_D^\bullet\mathcal{A}$.

5.5. To introduce a hermitian structure on $\Omega_D^\bullet C^\infty(M)$, we need a basic technical lemma.

Lemma 5.7. *Let $(\mathcal{H}, D)$ be an n^+-summable K-cycle on $\mathcal{A} = C^\infty(M)$. If $T \in \pi(\Omega^\bullet \mathcal{A})$ and S is a bounded operator on $\mathcal{H}$, then*

$$\mathrm{Tr}^+(ST|D|^{-n}) = \mathrm{Tr}^+(S|D|^{-n}T). \tag{5.15}$$

Proof. The statement is that $S(T|D|^{-n} - |D|^{-n}T) \in \mathcal{L}_0^{1+}(\mathcal{H})$; since S is bounded and $\mathcal{L}_0^{1+}$ is an ideal, we can take $S = 1$. Moreover, since

$$T|D|^{-n} - |D|^{-n}T = \sum_{r=0}^{n-1} |D|^{-r}(T|D|^{-1} - |D|^{-1}T)|D|^{-n+r+1},$$

we have just to show that $T|D|^{-1} - |D|^{-1}T \in \mathcal{L}_0^{n+}$. And it is enough to show this for $T = a$ or $T = [D, a]$ with $a \in A$.

Now multiplication by a, or Clifford multiplication by $\boldsymbol{d}a$, are pseudodifferential operators of order 0, whereas $|D|^{-1}$ is a pseudodifferential operator of order -1, its principal symbol being $\|\xi\|^{-1}$. From the symbol calculus formulae (2.4), if P and Q are of orders m and m', then $[P, Q]$ is of order $m + m' - 1$ at most, since the $m + m'$ term of its complete symbol clearly cancels. So $[|D|^{-1}, a]$ and $[|D|^{-1}, [D, a]]$, and more generally $[D, T]$ for $T \in \pi(\Omega^k \mathcal{A})$, is of order -2 at most. Therefore $T|D|^{-n} - |D|^{-n}T$ has order $(-n-1)$ at most, so by the trace theorem, its Dixmier trace vanishes. □

Remark. Connes asserts in ref. [10, p. 206] that the conclusion of lemma 5.7 holds for any n^+-summable K-cycle. This is plausible, but we are not aware of a proof at the moment. Therefore, we provisionally propose the following definition.

Definition. An n^+-summable K-cycle $(\mathcal{H}, D)$ on an algebra $\mathcal{A}$ is *tame* if for any $T \in \pi(\Omega^\bullet \mathcal{A})$ and $S \in \mathcal{L}(\mathcal{H})$, (5.15) holds.

From tameness and the traciality of Tr^+, the following three traces coincide and define an inner product on $\pi(\Omega^k \mathcal{A})$:

$$\langle S \mid T\rangle := \mathrm{Tr}^+(S^\dagger T|D|^{-n}) = \mathrm{Tr}^+(S^\dagger |D|^{-n}T) = \mathrm{Tr}^+(T|D|^{-n}S^\dagger). \tag{5.16}$$

Let us write $\widetilde{\mathcal{H}}_k$ for the Hilbert space obtained by completing $\pi(\Omega^k \mathcal{A})$ with respect to this inner product.

For $a \in \mathcal{A}$ and $S, T \in \pi(\Omega^k \mathcal{A})$ we have

$$\langle aS \mid aT\rangle = \mathrm{Tr}^+(T|D|^{-n}S^\dagger a^* a), \qquad \langle Sa \mid Ta\rangle = \mathrm{Tr}^+(aa^* S^\dagger |D|^{-n}T). \tag{5.17}$$

This says that the unitary group $\mathcal{U}(\mathcal{A}) := \{u \in \mathcal{A} : u^*u = uu^* = 1\}$ has two commuting unitary representations L and R on $\widetilde{\mathcal{H}}_k$, given by left and right multiplications. Now $\pi(dJ_0^{k-1})$ is a subbimodule of $\pi(\Omega^k \mathcal{A})$, by the obvious generalization of (5.9), so the unitary representations leave its closure in $\widetilde{\mathcal{H}}_k$

invariant. Let P be the orthogonal projector on $\widetilde{\mathcal{H}}_k$ whose range is the orthogonal complement of $\pi(dJ_0^{k-1})$, and define $\mathcal{H}_k := P\widetilde{\mathcal{H}}_k$. Then P commutes with $L(a)$ and $R(a)$ for $a \in \mathcal{U}(\mathcal{A})$ and so for any $a \in \mathcal{A}$; thus

$$P(aTa') = aP(T)a' \quad \text{for } T \in \widetilde{\mathcal{H}}_k,\ a, a' \in \mathcal{A}. \tag{5.18}$$

Now the projector P is just the continuous extension to $\widetilde{\mathcal{H}}_k$ of the quotient map from $\pi(\Omega^k\mathcal{A})$ to $\Omega_D^k\mathcal{A}$. Thus $\Omega_D^k\mathcal{A}$ is identified with a dense subspace of $\mathcal{H}_k$. Moreover, the left and right representations of $\mathcal{A}$ on $\widetilde{\mathcal{H}}_k$ reduce to algebra representations on $\mathcal{H}_k$, on account of (5.18), which extend the left and right module actions of $\mathcal{A}$ on $\Omega_D^k\mathcal{A}$.

Let us go back to $\Omega_D^\bullet C^\infty(M)$. From now on, in view of the application to the Standard Model, we will assume that the dimension n of M is *even*. If $T \in \Omega^k\mathcal{A}$ with $\pi(T) = c(s)$, we have $P\pi(T) = c(\eta)$ in $\mathcal{H}_k$, where η is the component of s in $\Gamma(C^k \ominus C^{k-1})$. The trace theorem now gives

$$\begin{aligned}
&\langle P\pi(T) \mid P\pi(T)\rangle^2_{\mathcal{H}_k} \\
&= \frac{1}{n\,(2\pi)^n} \int_{\mathbb{S}^*M} \operatorname{tr} \sigma_{-n}(c(\eta)^\dagger c(\eta)|D|^{-n}) \\
&= \frac{(4\pi)^{-n/2}}{\Gamma(\frac{1}{2}n+1)} \int_M \operatorname{tr}(c(\eta)^\dagger c(\eta))\,\mathrm{d}x \\
&= \frac{(2\pi)^{-n/2}}{(n/2)!} \int_M \eta \wedge *\eta \equiv \frac{(2\pi)^{-n/2}}{(n/2)!} \int_M \|\eta\|^2\,\mathrm{d}x.
\end{aligned} \tag{5.19}$$

The third equality follows from the formula for the trace of the spin representation (A.15); we recall that if the k-form η is written in local coordinates as $\eta = \sum_K a_K e_{k_1} \wedge \cdots \wedge e_{k_r}$ and $*\eta$ denotes its Hodge dual, then $\eta \wedge *\eta = (\sum_K |a_K|^2) e_1 \wedge \cdots \wedge e_n$, which thus equals $2^{-n/2}\operatorname{tr}(c(\eta)^\dagger c(\eta))$ times the volume form on M. We adopt the usual notation of writing this multiple as the squared norm of η. We conclude that the inner product on $\mathcal{H}_k$ given by (5.16) corresponds, under the identification σ_D, with the hermitian structure on $\mathcal{E}^\bullet(M)$ arising from the Riemannian metric.

5.6. We now turn to the example of the two-point space $\mathcal{A} = \mathbb{C}^2$, and construct K-cycles over this algebra.

Take $\mathcal{H} = \mathbb{C}^N \oplus \mathbb{C}^N$, with the action of $\mathcal{A} = \mathbb{C}^2$ given by $a(s_1, s_2) := (a_1 s_1, a_2 s_2)$. We denote by D the hermitian matrix

$$D := \begin{pmatrix} 0 & m^\dagger \\ m & 0 \end{pmatrix}, \tag{5.20}$$

where m is *any* nonzero $N \times N$ matrix. The grading is given by $\Gamma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

The diagonal action of $\mathcal{A}$ is even for this grading. One has

$$D^2 = \begin{pmatrix} m^\dagger m & 0 \\ 0 & mm^\dagger \end{pmatrix}. \tag{5.21}$$

We compute:

$$\mathrm{i}[D,a] = \mathrm{i}(a_1 - a_2)\begin{pmatrix} 0 & -m^\dagger \\ m & 0 \end{pmatrix}, \tag{5.22a}$$

$$-[D,a]^2 = (a_1 - a_2)^2\begin{pmatrix} m^\dagger m & 0 \\ 0 & mm^\dagger \end{pmatrix}, \tag{5.22b}$$

so $\|[D,a]\|^2 = |a_1 - a_2|^2 \|m^\dagger m\|$, which, together with the distance formula (5.3), yields $d(q_1, q_2) = 1/\|m^\dagger m\|^{1/2}$.

The K-cycle $(\mathcal{H}, D)$ is tame. Notice first that for a finite-dimensional Hilbert space, the Dixmier trace is merely a multiple of the usual trace; we therefore replace Tr^+ by Tr. Moreover, the dimension n of this K-cycle is zero, since the Dixmier ideal is $\mathcal{L}(\mathcal{H})$.

Suppose we have a skew form $\alpha = (\phi^* - 1, \phi - 1) = -(\phi^* - 1)p\,dp + (\phi - 1)(1-p)\,dp$ in $\Omega^1\mathbb{C}^2 = \mathbb{C}^2$. Since $p = (1,0)$, we have at once:

$$\pi(p) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi(dp) = \mathrm{i}\begin{pmatrix} 0 & -m^\dagger \\ m & 0 \end{pmatrix},$$

$$\pi(dp\,dp) = \begin{pmatrix} m^\dagger m & 0 \\ 0 & mm^\dagger \end{pmatrix}. \tag{5.23}$$

Therefore,

$$\pi(\alpha) = \begin{pmatrix} 0 & \mathrm{i}(\phi^* - 1)m^\dagger \\ \mathrm{i}(\phi - 1)m & 0 \end{pmatrix}. \tag{5.24}$$

The curvature of the universal connection $d + \alpha$ on $\mathcal{A}$ is $\theta = d\alpha + \alpha^2 = (2 - \phi - \phi^*)dp\,dp - |\phi - 1|^2 dp\,dp = (1 - |\phi|^2)dp\,dp$, which projects to

$$\pi(\theta) = (1 - |\phi|^2)\pi(dp\,dp) = (1 - |\phi|^2)D^2. \tag{5.25}$$

In this example π *is injective*, so $\pi(\Omega^\bullet\mathcal{A})$ and $\Omega_D^\bullet\mathcal{A}$ coincide.

6. Noncommutative geometry: the action

6.1. Recall that any skew form $\alpha \in \Omega^1\mathcal{A}$ determines a universal connection $\nabla = d + \alpha$ on the trivial bundle $\mathcal{E} = \mathcal{A}$, whose curvature is $\theta := d\alpha + \alpha^2$. Let $(\mathcal{H}, D)$ be a tame K-cycle on $\mathcal{A}$. We then define the *pre-Yang–Mills functional*:

$$I(\nabla) := \mathrm{Tr}^+(\pi(\theta)^2 |D|^{-n}). \tag{6.1}$$

Then $I(\nabla) \geq 0$, since it is the square of the norm of $\pi(\theta)$ in $\widetilde{\mathcal{H}}_2$, and moreover this functional is *gauge invariant*. To see that, recall that for any $u \in \mathcal{U}(\mathcal{A})$, the gauge transformation γ_u acts on the curvature θ by $\gamma_u(\theta) = u\theta u^*$, and so by tameness

$$\begin{aligned} I(\gamma_u(\nabla)) &= \mathrm{Tr}^+(\pi(\theta)^2 u^*|D|^{-n}u) \\ &= \mathrm{Tr}^+(\pi(\theta)^2 u^*u|D|^{-n}) = I(\nabla). \end{aligned} \tag{6.2}$$

6.2. In the commutative Riemannian case, using the Dirac K-cycle, let ω be a one-form in $\mathcal{E}^1(M)$, and choose $\alpha \in \Omega^1\mathcal{A}$ with $\pi_D(\alpha) = \omega$; then $\sigma_2(\pi(d\alpha)) = \mathrm{i}\,\boldsymbol{d}\omega$. For any two such α, the corresponding operators $\pi(d\alpha)$ differ by an element of $\pi(dJ_0^1) = c(\ker\sigma_2)$.

We thus get $P(\pi(d\alpha + \alpha^2)) = \mathrm{i}\,\boldsymbol{d}\omega$ in $\widetilde{\mathcal{H}}_2$. From the nearest-point property of orthogonal projectors, we thus have

$$\inf\{I(\nabla) : \pi_D(\alpha) = \omega\} = \langle \boldsymbol{d}\omega \mid \boldsymbol{d}\omega\rangle_2 = \frac{(2\pi)^{-n/2}}{(n/2)!}\int_M \|\boldsymbol{d}\omega\|^2\,\mathrm{d}x. \tag{6.3}$$

Let us call the right hand side of this equation $\mathrm{YM}(\nabla)$. Since the curvature of the classical connection determined by ω on the trivial line bundle over M is just $\boldsymbol{d}\omega$, we see that $\mathrm{YM}(\nabla)$ is, in this instance, just the usual Yang–Mills action of classical gauge theory. The trace theorem can now be regarded as saying that the Dixmier trace is the "dequantizer" for the K-cycles built over the setting of classical gauge theories. Notice that for $n = 4$, the constant in $\mathrm{YM}(\nabla)$ is just $1/8\pi^2$.

6.3. The previous discussion tells us that the correct definition of the Yang–Mills action in noncommutative geometry is given by $\langle d\omega + \omega^2 \mid d\omega + \omega^2\rangle$, where ω is a one-form and the derivation and inner product are taken in the sense of $\Omega_D\mathcal{A}$. To determine it for an arbitrary hermitian vector bundle $\mathcal{E} = p\mathcal{A}^m$, we introduce connections whose values are D-forms. Such a connection is a linear map $\widetilde{\nabla}: \mathcal{E} \to \mathcal{E}\otimes_{\mathcal{A}}\Omega_D^1\mathcal{A}$ satisfying the analogue of (4.34) with the derivation taken in $\Omega_D\mathcal{A}$. From the universal property of $\Omega^1\mathcal{A}$, there is a universal connection ∇ such that $(\mathrm{id}\otimes\pi_D)(\nabla s) = \mathrm{i}\widetilde{\nabla}s$ for $s \in \mathcal{E}$.

The curvature of $\widetilde{\nabla}$ is $\theta^{\widetilde{\nabla}} := \widetilde{\nabla}^2 \in \mathrm{End}_{\mathcal{A}}(\mathcal{E}, \mathcal{E}\otimes_{\mathcal{A}}\Omega_D^2\mathcal{A})$. We have $\theta^{\widetilde{\nabla}} = -(\mathrm{id}\otimes\pi_D)(\theta)$, where θ is the curvature of ∇, and we define

$$\mathrm{YM}(\widetilde{\nabla}) := \langle \theta^{\widetilde{\nabla}} \mid \theta^{\widetilde{\nabla}}\rangle, \tag{6.4}$$

where this inner product incorporates the one induced on $\mathrm{End}_{\mathcal{A}}\mathcal{E}$ by the hermitian structure on $\mathcal{E}$. To make this precise, we remark that the tensor product $\mathcal{E}\otimes_{\mathcal{A}}\mathcal{H}$ can be made a Hilbert space with inner product

$$\langle s_1\otimes\eta_1 \mid s_2\otimes\eta_2\rangle := \langle \eta_1 \mid (s_1 \mid s_2)\eta_2\rangle, \quad s_1, s_2 \in \mathcal{E};\ \eta_1, \eta_2 \in \mathcal{H}. \tag{6.5}$$

Here $\langle\eta_1 \mid \eta_2\rangle$ denotes the inner product in $\mathcal{H}$ and the action of $\mathcal{A}$ on $\mathcal{H}$ is understood. Henceforth we will write simply π (rather than $\mathrm{id}\otimes\pi$) for the homomorphism from $\mathcal{E}\otimes_{\mathcal{A}}\Omega^\bullet\mathcal{A}$ to $\mathcal{L}(\mathcal{E}\otimes_{\mathcal{A}}\mathcal{H})$ extending (5.6). Now (5.16), with D

replaced by $1 \otimes D$, defines an inner product on $\pi(\mathcal{E} \otimes_{\mathcal{A}} \Omega^k \mathcal{A})$. As before, one has an orthogonal projector P whose range may be identified with the completion of $\mathcal{E} \otimes_{\mathcal{A}} \Omega_D^2 \mathcal{A}$; adapting previous arguments, we arrive at

$$\mathrm{YM}(\widetilde{\nabla}) = \inf\{ I(\nabla) : \pi_D(\nabla) = \mathrm{i}\widetilde{\nabla} \} = \langle P\pi(\theta) \mid P\pi(\theta) \rangle. \tag{6.6}$$

For the Dirac K-cycle on M, this is computed by (5.19), where now $\eta = \mathrm{i}(\boldsymbol{d}\omega + \omega \wedge \omega)$ is the matrix-valued two-form such that $P\pi(\theta) = P\pi(d\alpha + \alpha^2) = c(\eta)$ when $\nabla = pd + \alpha$ and $\pi(\alpha) = \omega$. We thereby recover the standard Yang–Mills functional on general vector bundles:

$$\mathrm{YM}(\widetilde{\nabla}) = \frac{(2\pi)^{-n/2}}{(n/2)!} \int_M \|\boldsymbol{d}\omega + \omega \wedge \omega\|^2 \,\mathrm{d}x. \tag{6.7}$$

6.4. In many cases there is a curvature-independent lower bound for $I(\nabla)$ which arises from a Hochschild cocycle corresponding, in the commutative Riemannian case, to the fundamental homology class of the underlying manifold. To glimpse this truth, we introduce the following Hochschild cochain:

$$\varphi(a_0, \dots, a_n) := \mathrm{Tr}^+(\Gamma a_0 [D, a_1] \cdots [D, a_n] D^{-n}), \tag{6.8}$$

where $(\mathcal{H}, D)$ is a tame K-cycle over $\mathcal{A}$ and Γ is the grading operator on $\mathcal{H}$. This φ is in fact a Hochschild cocycle: here (4.27) telescopes to

$$\begin{aligned} b\varphi(a_0, \dots, a_{n+1}) = \{ &\mathrm{Tr}^+(\Gamma a_0 [D, a_1] \cdots [D, a_n]\, a_{n+1} D^{-n}) \\ &- \mathrm{Tr}^+(\Gamma a_{n+1} a_0 [D, a_1] \cdots [D, a_n] D^{-n}) \} = 0, \end{aligned} \tag{6.9}$$

since Tr^+ is a trace, and the tameness allows a_{n+1} to slip past D^{-n} under the Dixmier trace. One can check that $\lambda\varphi \neq \varphi$ in general, so φ need not be a *cyclic* cocycle. However, it turns out that φ is cyclic in many cases.

For example, in the case of the Dirac K-cycle, we get from the trace theorem:

$$\begin{aligned} \varphi(a_0, \dots, a_n) &= \frac{1}{n\,(2\pi)^n} \int_{\mathbb{S}^* M} \mathrm{tr}\, \sigma_{-n}(c(\gamma) c(a_0 \boldsymbol{d}a_1 \cdots \boldsymbol{d}a_n) D^{-n}) \\ &= \frac{(2\pi)^{-n/2}}{(n/2)!} \int_M a_0\, \boldsymbol{d}a_1 \wedge \cdots \wedge \boldsymbol{d}a_n, \end{aligned} \tag{6.10}$$

where γ is the chirality section (A.13) of the Clifford bundle, since $\Gamma = c(\gamma)$, and since $\mathrm{tr}(c(\gamma) c(a_0 \boldsymbol{d}a_1 \cdots \boldsymbol{d}a_n))$ times the volume form on M equals $a_0\, \boldsymbol{d}a_1 \wedge \cdots \wedge \boldsymbol{d}a_n$, by (A.15). Note that for this example, φ is indeed a cyclic cocycle.

This suggests that the cohomology class $[\varphi] \in H^n(\mathcal{A}, \mathcal{A}')$ is the right generalization of the notion of "fundamental class" of a Riemannian manifold in noncommutative geometry. As it happens, this is not the whole story, since a further subtlety is involved: one may in many cases deform φ to a cohomologous Hochschild cocycle τ, given by $\tau(a_0, \dots, a_n) := \mathrm{Tr}(\Gamma F [F, a_0] \cdots [F, a_n])$ with

$F = D|D|^{-1}$; now τ is always a cyclic cocycle, and it is the class $[\tau] \in HC^n(\mathcal{A})$ which provides the right generalization [10].

To see that φ provides a lower bound for $I(\nabla)$, we consider first the case $\mathcal{E} = \mathcal{A}$, and take $n = 4$. In this case, using (4.28), $I(\nabla) = \hat{\psi}(\theta^2)$, where

$$\psi(a_0, \dots, a_4) \equiv \hat{\psi}(a_0\, da_1 \cdots da_4) := \mathrm{Tr}^+(a_0[D, a_1] \cdots [D, a_4]D^{-4}), \quad (6.11)$$

so that ψ is also a Hochschild cocycle, as is seen by dropping Γ from (6.9). Now, the sum and difference $\psi \pm \varphi$ are *positive* cocycles, i.e.,

$$\begin{aligned}(\hat{\psi} \pm \hat{\varphi})(a_0\, da_1\, da_2(a_0\, da_1\, da_2)^*) &= (\hat{\psi} \pm \hat{\varphi})(a_0\, da_1\, da_2\, da_2^*\, da_1^*\, a_0^*) \\ &= \mathrm{Tr}^+(a_0[D, a_1][D, a_2]D^{-2}(1 \pm \Gamma)D^{-2}(a_0[D, a_1][D, a_2])^*) \geq 0, \quad (6.12)\end{aligned}$$

since $\frac{1}{2}(1 \pm \Gamma)$ are positive operators (in fact, orthogonal projectors) commuting with D^{-2} and with $\pi(\Omega^2\mathcal{A})$. The inequality shows that $\langle\omega|\eta\rangle_\pm := (\hat{\psi} \pm \hat{\varphi})(\eta\omega^*)$ is a positive inner product on $\Omega^2\mathcal{A}$. In particular, since $\theta^* = \theta$, we arrive at the following inequality:

$$I(\tilde{\nabla}) = \hat{\psi}(\theta^2) \geq |\hat{\varphi}(\theta^2)| = |\mathrm{Tr}^+(\Gamma\pi(\theta)^2 D^{-4})|. \quad (6.13)$$

Note that the same argument works whenever n is divisible by 4 (since we require that $D^{-n/2}$ be an even operator).

To extend (6.13) to the general case where $\mathcal{E} = p\mathcal{A}^m$, we assume that φ is a cyclic cocycle, so that $\hat{\varphi}$ is a trace on $\Omega^2\mathcal{A}$ on account of (4.33); we know that $\hat{\psi}$ is also a trace there since $(\mathcal{H}, D)$ is tame. By tensoring these with the matrix trace on $\mathrm{End}_{\mathcal{A}}(p\mathcal{A}^m)$, we prolong them to traces on $\mathrm{End}_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega^2\mathcal{A})$, which we continue to denote by $\hat{\varphi}$ and $\hat{\psi}$. One checks that the inequality $I(\tilde{\nabla}) = \hat{\psi}(\theta^2) \geq |\hat{\varphi}(\theta^2)|$ remains valid in the general case.

Finally, we show that the right hand side $|\hat{\varphi}(\theta^2)|$ is independent of the universal connection $\tilde{\nabla}$, on account of the cyclicity of φ. Indeed, if $\tilde{\nabla} = pd + \alpha$, write $\tilde{\nabla}_t := pd + t\alpha$ for $0 \leq t \leq 1$. Then

$$\begin{aligned}(\mathrm{d}/\mathrm{d}t)\big|_{t=0}\hat{\varphi}(\theta_t^2) &= (\mathrm{d}/\mathrm{d}t)\big|_{t=0}\hat{\varphi}((p\, dp\, dp + tp\, d\alpha\, p + t^2\alpha^2)^2) \\ &= 2\hat{\varphi}(p\, dp\, dp\, d\alpha\, p) = 2\hat{\varphi}(dp\, dp\, p\, d\alpha) \\ &= 2\hat{\varphi}(dp\, dp\, d\alpha) - 2\hat{\varphi}((dp)^3\alpha) \\ &= 2\hat{\varphi} \circ d(p\, dp\, d\alpha) - 2\hat{\varphi}(p(dp)^3 p\alpha) = 0, \quad (6.14)\end{aligned}$$

using the traciality of $\hat{\varphi}$, the property $\hat{\varphi} \circ d = 0$ of the definition (6.8), the identity $\alpha = p\alpha p$, and $p(dp)^3 p = 0$ from (4.17). In consequence, we have $\hat{\varphi}(\theta^2) = \hat{\varphi}((p\, dp\, dp)^2) = \varphi(p, p, p, p, p)$.

We have finally the desired lower bound $\mathrm{YM}(\nabla) \geq |\varphi(p, p, p, p, p)|$ for $n = 4$. The fine theory of cyclic cohomology allows one to say much more about the right hand side. It turns out [7] that it depends only on the stable isomorphism class $[p] \in K_0(\mathcal{A})$ of the vector bundle $\mathcal{E}$ and on the cyclic cohomology class $[\varphi] \in HC^4(\mathcal{A})$; for the Dirac case, with $\mathcal{E} = \Gamma(E)$, it is related via the index theorem to the Chern classes of the ordinary vector bundle E. Whenever one can show this term to be positive, one has a measure of the "nonflatness" of the underlying gauge theory.

6.5. To the Yang–Mills action we now want to add a "fermionic piece". Suppose we have a tame K-cycle $(\mathcal{H}, D)$ on $\mathcal{A}$ and a vector bundle $\mathcal{E}$ over $\mathcal{A}$; let $\tilde{\nabla}$ be a compatible connection with values in $\mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D \mathcal{A}$. On the Hilbert space $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ with inner product (6.5), we want to define a self-adjoint operator $D_{\tilde{\nabla}}$ by the minimal coupling recipe:

$$D_{\tilde{\nabla}}(s \otimes \eta) := s \otimes D\eta + (\tilde{\nabla} s)\eta. \tag{6.15}$$

At first glance this is not well defined, since $\tilde{\nabla} s$ is not, strictly speaking, an operator on $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ on account of the quotient involved in the definition (5.7) of $\Omega^1_D \mathcal{A}$. Thus we take a universal connection ∇ for which $\pi_D(\nabla) = \mathrm{i}\tilde{\nabla}$, and define $D_{\tilde{\nabla}}$ as

$$D_{\tilde{\nabla}}(s \otimes \eta) := s \otimes D\eta - \mathrm{i}\pi(\nabla s)\eta. \tag{6.16}$$

If $\mathcal{E} = p\mathcal{A}^m$ and $\nabla = pd + \alpha$, then we have $D_{\tilde{\nabla}} = pD - \mathrm{i}\pi(\alpha)$, where we regard D as acting componentwise on $\mathcal{A}^m \otimes_{\mathcal{A}} \mathcal{H}$. This is clearly a symmetric operator on $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$, since $\pi(\alpha)$ is skew-adjoint; so it is a self-adjoint operator with domain $\mathcal{E} \otimes_{\mathcal{A}} \mathrm{Dom}(D)$. Moreover, two universal connections with image $\mathrm{i}\tilde{\nabla}$ differ by $\alpha_1 - \alpha_2 \in \ker \pi$, so the right hand side of (6.16) depends only on $\tilde{\nabla}$.

The fermionic action is now given by

$$I_F(\psi) := \langle \psi \mid D_{\tilde{\nabla}} \psi \rangle \tag{6.17}$$

for "wavefunctions" $\psi \in \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$.

Lemma 6.1. *This action is gauge invariant, i.e.,* $\langle u\psi \mid D_{\gamma_u(\tilde{\nabla})} u\psi \rangle = I_F(\psi)$ *for* $u \in \mathcal{U}(\mathcal{E})$.

Proof. It suffices to show that $uD_{\tilde{\nabla}}u^* = D_{\gamma_u(\tilde{\nabla})}$ on $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$. If $x \in u\,\mathrm{Dom}\, D_{\tilde{\nabla}}$, we have

$$\begin{aligned} uD_{\tilde{\nabla}}u^* x &= u(pD - \mathrm{i}\pi(\alpha))u^* x \\ &= puD(u^* x) - \mathrm{i}u\pi(\alpha)u^* x \\ &= pDx + u(Du^*)x - \mathrm{i}\pi(u\alpha u^*)x \\ &= pDx - \mathrm{i}\pi(u\,du^* + u\alpha u^*)x \\ &= (pD - \mathrm{i}\pi(\gamma_u(\alpha)))x, \end{aligned} \tag{6.18}$$

using (4.41). □

With this, we are nearing the construction of a dictionary spelling out the translation between noncommutative geometry and particle physics. The Hilbert space $\mathcal{H}$ and the Dirac operator D have the more straightforward translation; the first is the Hilbert space of fermions. They are defined now in a compact "Euclidean" space, rather than in Minkowskian spacetime, but never mind. Then D is plainly the Dirac operator familiar from quantum electrodynamics. The algebra $\mathcal{A}$ and the bundle $\mathcal{E}$ are related to gauge transformations, and the Yang–Mills action corresponds to the pure gauge boson part of the action in particle theory. The obtained action must be "Wick-rotated" to Minkowski space. *After* that process leading to a Poincaré (and gauge) invariant action, we will impose the chirality condition on our fermions. Thus, the concept of K-cycle is an embodiment of the "neutrino paradigm" that pervades modern particle physics [33].

6.6. Other ingredients, like the Higgs fields that result from "spontaneous symmetry breaking" in the standard approach, giving rise to mass for some gauge bosons, the Yukawa couplings giving mass to fermions... are already present in schematic form in the example of the two-point space.

Example. We come back to gauge theory on the finite space $\{q_1, q_2\}$, but in a slightly less trivial context: we consider the simplest nontrivial bundle of rank two over q_1 and rank one over q_2.

First we briefly reexamine the trivial rank-one bundle $\mathcal{E} = \mathcal{A}$, with the tame K-cycle $(\mathbb{C}^N \oplus \mathbb{C}^N, D)$ given by (5.20), and the connection $\nabla = d + \alpha$ with $\alpha = (1-\phi^*)p\,dp + (\phi-1)(1-p)\,dp$; recall that $\pi(\theta) = (1-|\phi|^2)D^2$ by (5.25). Since $n = 0$ and $\mathrm{Tr}^+ = \mathrm{Tr}$ for this finite dimensional example, we have

$$\mathrm{YM}(\nabla) = \mathrm{Tr}(\pi(\theta)^2) = 2(1-|\phi|^2)^2\,\mathrm{tr}((m^\dagger m)^2). \tag{6.19}$$

Recall that the gauge group $\mathrm{U}(1) \times \mathrm{U}(1)$ acts by $\phi \mapsto u_1^* u_2 \phi$, so that (6.19) is manifestly gauge invariant. Now

$$D_\nabla = D - \mathrm{i}\pi(\alpha) = \begin{pmatrix} 0 & \phi^* m^\dagger \\ \phi m & 0 \end{pmatrix}, \tag{6.20}$$

the fermionic action being $\langle \psi \,|\, D_\nabla \psi \rangle = 2\,\mathrm{Re}(\phi \psi_2^* m \psi_1)$ for $\psi \in \mathcal{H}$. As indicated above, we have already reproduced here the situation of symmetry breaking (the ground state of the Yang–Mills action, given here by $|\phi| = 1$, is nonunique and acted upon nontrivially by the gauge group) and the situation of "Yukawa coupling" between the "fields" ϕ and ψ.

Now we consider the vector bundle $\mathcal{E} = f\mathcal{A}^2$, where $f := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1-p \end{smallmatrix}\right)$. A connection is of the form $\nabla s = f\,ds + \alpha s$ with $\alpha = f\alpha f = -\alpha^*$. If we write $\alpha = ap\,dp - a^*(1-p)\,dp$ with $a \in \mathbb{C}^{2\times 2}$, the condition $\alpha = f\alpha f$ amounts to $a_{21} = a_{22} = 0$. Let us write $a_{11} =: 1 - \phi_1^*$, $a_{12} =: -\phi_2^*$, so that

$$\alpha = \begin{pmatrix} (1-\phi_1^*)p\,dp + (\phi_1 - 1)(1-p)\,dp & -\phi_2^* p\,dp \\ -\phi_2(1-p)\,dp & 0 \end{pmatrix}. \tag{6.21}$$

Then the curvature θ is $f\,df\,df + f\,d\alpha\,f + \alpha^2$, that is,

$$\theta = \begin{pmatrix} (1-|\phi_1|^2)dp\,dp - |\phi_2|^2 p\,dp\,dp & -\phi_1\phi_2^*(1-p)\,dp\,dp \\ -\phi_1^*\phi_2(1-p)\,dp\,dp & (1-|\phi_2|^2)(1-p)\,dp\,dp \end{pmatrix}, \tag{6.22}$$

so the diagonal entries of θ^2 are $(1-|\phi_1|^2-|\phi_2|^2)^2 p(dp)^4 + ((1-|\phi_1|^2)^2 + |\phi_1\phi_2|^2)\,(1-p)(dp)^4$ and $((1-|\phi_2|^2)^2 + |\phi_1\phi_2|^2)\,(1-p)(dp)^4$. Now, bearing in mind (5.23), we compute $\mathrm{Tr}(\pi(\theta)^2)$. Since $\mathrm{Tr}\,\pi(p(dp)^4) = \mathrm{Tr}\,\pi((1-p)(dp)^4) = \mathrm{tr}((m^\dagger m)^2)$, we get

$$\mathrm{YM}(\nabla) = \mathrm{Tr}(\pi(\theta^2)) = (1 + 2(1-|\phi_1|^2-|\phi_2|^2)^2)\;\mathrm{tr}((m^\dagger m)^2). \tag{6.23}$$

This is by construction invariant under the gauge group $\mathrm{U}(2)\times\mathrm{U}(1)$, which spells some trouble, as we shall need $\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge invariance; but let us not anticipate events. The space of sections realizing the minimum of the Yang–Mills action is a three-sphere; note that this minimum is now positive, so the bundle is not flat. One easily checks that for this example $\hat\varphi(f(df)^4) = -\,\mathrm{tr}((m^\dagger m)^2)$, so the minimum of (6.23) is given by the general estimate obtained in (6.13).

6.7. It is a feature of the formulation of the Standard Model in noncommutative geometry that the K-cycle one needs is not just a module over one algebra, but a bimodule over two. Very roughly speaking (we will later be more precise), one algebra incorporates the electroweak gauge group and the colour symmetries belong to the other. To see that it is perfectly natural, within the mathematical framework of noncommutative geometry, to consider two commuting algebras acting on the same Hilbert space, we pause to examine how a version of *Poincaré duality* may be formulated in the noncommutative case.

By Poincaré duality (over an ordinary compact n-dimensional Riemannian manifold M) we understand the isomorphism of de Rham differential forms $\omega \mapsto *\omega : \mathcal{E}^k(M) \to \mathcal{E}^{n-k}(M)$ determined by $\int_M \eta\wedge *\omega = \int_M (\eta\mid\omega)\,\mathrm{d\,vol}$, where the pairing of k-forms $(\eta\mid\omega)$ is given by the metric on M. For a fixed η, the continuous linear form $*\omega \mapsto \int_M (\eta\mid\omega)\,\mathrm{d\,vol}$ thereby determines a de Rham current C_η, which in turn determines a unique Hochschild cohomology class in $H^{n-k}(\mathcal{A},\mathcal{A}')$; here $\mathcal{A} = C^\infty(M)$. Recalling that the algebra of D-forms for the Dirac K-cycle is just the de Rham algebra, we thereby get a canonical map:

$$\Omega_D^k(\mathcal{A}) \to H^{n-k}(\mathcal{A},\mathcal{A}'). \tag{6.24}$$

The point at issue is that the same algebra appears on both sides of (6.24) only if $\mathcal{A}$ is commutative. In general the right hand side must be replaced by $H^{n-k}(\mathcal{B},\mathcal{B}')$, where $\mathcal{B}$ is a new algebra. Below we will give some sufficient conditions for a suitable choice of $\mathcal{B}$. On a deeper level, the right "Poincaré dual algebra" may be sought systematically using the KK-theory of Kasparov: this is discussed in ref. [13].

6.8. Let us recall that if $\mathcal{A}$, $\mathcal{B}$ are two algebras, a representation ρ of the tensor product $\mathcal{A} \otimes \mathcal{B}$ on a Hilbert space $\mathcal{H}$ is given whenever there are commuting representations ρ_1 of $\mathcal{A}$ and ρ_2 of $\mathcal{B}$ on $\mathcal{H}$; one has $\rho(a \otimes b) = \rho_1(a)\rho_2(b)$.

Connes [13] has identified the conditions of the following definition as being the main ingredients of Poincaré duality in noncommutative geometry.

Definition. Let $\mathcal{A}, \mathcal{B}$ be two algebras. We shall say that a graded K-cycle $(\mathcal{H}, D, \Gamma)$ over their tensor product is a *matching K-cycle* for $\mathcal{A} \otimes \mathcal{B}$ provided that:

(a) $(\mathcal{H}, D)$ is n^+-summable for some integer n, and is tame;

(b) for any $a \in \mathcal{A}$, $b \in \mathcal{B}$, we have $[[D,a],b] = 0$;

(c) the Hochschild cocycle φ for this K-cycle satisfies $\varphi(1, c_1, \dots, c_n) = 0$, i.e.,

$$\operatorname{Tr}^+(\Gamma[D,c_1]\cdots[D,c_n]|D|^{-n}) = 0, \quad \text{for } c_1, \dots, c_n \in \mathcal{A} \otimes \mathcal{B}. \tag{6.25}$$

Since $[[D,a],b] + [a,[D,b]] = [D,[a,b]] = 0$, we see that condition (b) is equivalent to $[[D,b],a] = 0$ for all $a \in \mathcal{A}$, $b \in \mathcal{B}$.

Consider the Dirac K-cycle $(L^2(S), D)$ over $\mathcal{A} = C^\infty(M)$. We can regard $\mathcal{H} = L^2(S)$ as a representation space for the algebra $\mathcal{A} \otimes \mathcal{A}$ (with both copies of $\mathcal{A}$ acting by multiplication operators), and so $(\mathcal{H}, D)$ is also a K-cycle over the algebra $\mathcal{A} \otimes \mathcal{A}$. This K-cycle satisfies (a) by lemma 5.7; (b) is just the observation that $c(\boldsymbol{d}a)$ commutes with b on the space of spinors; and (c) follows from (6.10) and Stokes' theorem.

The duality map is now given by the following result.

Theorem 6.2. *Suppose $(\mathcal{H}, D, \Gamma)$ is a matching K-cycle over the algebra $\mathcal{A} \otimes \mathcal{B}$. Then for each $\alpha \in \Omega^k(\mathcal{A})$, there is a Hochschild cocycle $c_\alpha \in Z^{n-k}(\mathcal{B}, \mathcal{B}')$ given by:*

$$c_\alpha(b_0, \dots, b_{n-k}) := (-)^{n-k} \operatorname{Tr}^+(\Gamma\pi(\alpha)\, b_0[D,b_1]\cdots[D,b_{n-k}]|D|^{-n}). \tag{6.26}$$

Moreover, c_α depends only on $\pi_D(\alpha)$, so $\pi_D(\alpha) \mapsto [c_\alpha]$ is a well-defined linear map from $\Omega^k_D\mathcal{A}$ to $H^{n-k}(\mathcal{B}, \mathcal{B}')$.

Proof. To show that c_α is a cocycle, we compute $bc_\alpha(b_0, \dots, b_{n-k+1})$. As in (6.9), this expression telescopes to

$$\begin{aligned} &\operatorname{Tr}^+(\Gamma\alpha\, b_0[D,b_1]\cdots[D,b_{n-k}]b_{n-k+1}|D|^{-n}) \\ &\quad - \operatorname{Tr}^+(\Gamma\alpha\, b_{n-k+1}b_0[D,b_1]\cdots[D,b_{n-k}]|D|^{-n}) = 0, \end{aligned} \tag{6.27}$$

using tameness and condition (b) to interchange b_{n-k+1} successively with $|D|^{-n}$, Γ, and $\pi(\alpha)$.

If $\alpha' = \sum_j a_1^j\, da_2^j \cdots da_k^j \in J^{k-1}$, then

$$\begin{aligned} c_{d\alpha'}&(b_0,\dots,b_{n-k}) \\ &= \mathrm{Tr}^+ (\Gamma b_0 \pi(d\alpha')\, [D,b_1] \cdots [D,b_{n-k}] |D|^{-n}) \\ &= \textstyle\sum_j \mathrm{Tr}^+ (\Gamma [D,b_0 a_1^j]\, [D,a_2^j] \cdots [D,a_k^j]\, [D,b_1] \cdots [D,b_{n-k}] |D|^{-n}) \\ &\quad - \mathrm{Tr}^+ (\Gamma [D,b_0] \pi(\alpha')\, [D,b_1] \cdots [D,b_{n-k}] |D|^{-n}), \end{aligned} \tag{6.28}$$

and the right hand side vanishes, using $\pi(\alpha') = 0$ and (6.25). Thus $c_\alpha = 0$ whenever $\pi_D(\alpha) = 0$; so c_α depends only on $\pi_D(\alpha)$ and gives a well-defined map from $\Omega_D^k(\mathcal{A})$ to $Z^{n-k}(\mathcal{B},\mathcal{B}')$. Passing to the quotient, we obtain the aforementioned map from $\Omega_D^k\mathcal{A}$ to $H^{n-k}(\mathcal{B},\mathcal{B}')$. □

Remark. The story can be taken further. If $d\alpha = 0$ in $\Omega_D^{k+1}\mathcal{A}$, one can show that c_α is a cyclic cocycle. A finer analysis then produces a linear map from the cohomology of the differential algebra $(\Omega_D^\bullet\mathcal{A}, d)$ to a quotient of the cyclic cohomology of the algebra $\mathcal{B}$. These algebraic developments are outlined in ref. [13]. The upshot is that Poincaré duality can be given a fully cohomological formulation in noncommutative geometry.

6.9. A necessary last step before turning to the Standard Model is a brief discussion of product spaces. Let $(\mathcal{H}_1, D_1)$, $(\mathcal{H}_2, D_2)$ be K-cycles on the respective algebras $\mathcal{A}_1, \mathcal{A}_2$, and let Γ_1 denote the grading operator on $\mathcal{H}_1$. Their *product* is the K-cycle $(\mathcal{H}_1 \otimes \mathcal{H}_2, D)$ for the algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$, where $D := D_1 \otimes 1 + \Gamma_1 \otimes D_2$. The trick of this definition is that $D^2 = D_1^2 \otimes 1 + 1 \otimes D_2^2$, showing that the orders of summability add up.

There is a canonical bimodule homomorphism from the space of forms $\Omega(\mathcal{A}_1 \otimes \mathcal{A}_2)$ to $\Omega\mathcal{A}_1 \otimes \Omega\mathcal{A}_2$. Given hermitian vector bundles $\mathcal{E}_1, \mathcal{E}_2$ associated respectively to $\mathcal{A}_1$, $\mathcal{A}_2$, there is a hermitian vector bundle $\mathcal{E}_1 \otimes \mathcal{E}_2$ associated to $\mathcal{A}_1 \otimes \mathcal{A}_2$.

7. The Glashow–Weinberg–Salam model from noncommutative geometry

7.1. We now have all the ingredients necessary to reconstruct the Standard Model. The contention is that *a pure gauge field with a fermionic current is able to give us all the intricacies of the standard model Lagrangian*, if we suitably modify the spacetime continuum. We next spell out in detail how the GWS Lagrangian is recovered, in a Euclidean spacetime framework with a noncommutative geometry. In this section and the next, we mainly follow Connes and Lott [11,12], incorporating the simplifications due to the π_D homomorphism brought in by Connes in ref. [13], described in section 5.

The general strategy will be to think of the algebra $\mathcal{A}$ together with a vector bundle $\mathcal{E}$ as specifying the gauge group, and the action of $\mathcal{A}$ on the Hilbert space as specifying the fermionic representation of the gauge group. Thus, as

$\mathcal{E} \simeq \mathbb{C}^n \otimes C^\infty(M)$, roughly speaking, and $\mathcal{H} \simeq L^2(S) \otimes \mathbb{C}^{N_G}$, the total space of fermions is $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \simeq \mathbb{C}^n \otimes L^2(S) \otimes \mathbb{C}^{N_G}$. We have denoted by N_G the number of leptonic generations. In other words, we obtain N_G fermions in the fundamental representation of the vector bundle.

To obtain the GWS model, we take the structure group $\mathrm{U}(1) \times \mathrm{U}(2)$ for the time being. We take the product, in noncommutative geometry, of the K-cycles corresponding to the four-dimensional space example and the two-point example, so the spacetime is formed by two copies of a compact spin^c manifold; we take a $\mathbb{C}$-bundle on one leaf and a $\mathbb{C}^2$-bundle on the other.

Recall that the two graded K-cycles we have been considering so far are the Dirac K-cycle $(L^2(S), \partial\!\!\!/, \gamma_5)$ over the algebra $C^\infty(M)$ and the K-cycle $(\mathbb{C}^{N_G} \oplus \mathbb{C}^{N_G}, D_m, \sigma_3)$ over the algebra $\mathbb{C} \oplus \mathbb{C}$. Here we abbreviate $\partial\!\!\!/ := \gamma^\mu \partial_\mu$ as usual, and

$$D_m := \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{7.1}$$

We assume here that m is a positive-definite matrix, indeed a real positive one, since we will interpret it as a mass matrix (the reality condition is not a restriction, as we may assume it to be diagonalized). For the product K-cycle over $\mathcal{A} = C^\infty(M) \oplus C^\infty(M)$, the operator can be written as

$$D := \partial\!\!\!/ \otimes (I \oplus I) + \gamma_5 \otimes D_m = \begin{pmatrix} \partial\!\!\!/ \otimes I & \gamma_5 \otimes m \\ \gamma_5 \otimes m & \partial\!\!\!/ \otimes I \end{pmatrix}. \tag{7.2}$$

This product K-cycle is 4^+-summable and tame.

As the vector bundle, we take the product of the trivial bundle $C^\infty(M)$ and the bundle $f((\mathbb{C}^2)^2)$, with $f := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1-p \end{smallmatrix}\right)$. Thus $\mathcal{E} = f((C^\infty(M) \otimes \mathbb{C}^2)^2) \simeq C^\infty(M) \otimes \mathbb{C}^3$. Note that $\mathcal{E} = \mathcal{A} \oplus (1-p)\mathcal{A}$. The total fermion space $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ is then $L^2(S) \otimes \mathbb{C}^3 \otimes \mathbb{C}^{N_G}$; an element of it can be written in the suggestive form

$$\psi := \begin{pmatrix} e_\mathrm{R} \\ e_\mathrm{L} \\ \nu_\mathrm{L} \end{pmatrix}, \tag{7.3}$$

where the entries live in $L^2(S) \otimes \mathbb{C}^{N_G}$; we shall write operators on this space as 3×3 matrices of operators over $L^2(S) \otimes \mathbb{C}^{N_G}$. (The presence of the projector f may be accounted for if one thinks of these as 4×4 matrices of operators with zero third row and column, which we will suppress.)

7.2. A universal connection ∇ is given by $\nabla s = f\,ds + \alpha s$, with $\alpha \in \mathcal{A}^{2\times 2} \otimes_{\mathcal{A}} \Omega_1\mathcal{A}$. Here α is a 2×2 matrix of one-forms, satisfying $f\alpha f = \alpha$ and $\alpha^* = -\alpha$.

To describe α more explicitly, we consider the structure of $\Omega^1\mathcal{A}$. Let $a = (a_1, a_2) \in \mathcal{A}$, with each $a_i \in C^\infty(M)$. An element $b \in \Omega^1\mathcal{A} \subset \mathcal{A} \otimes \mathcal{A}$ is given by a quadruple of functions $b = (b_{11}, b_{12}, b_{21}, b_{22})$ with each $b_{ij} \in C^\infty(M \times M)$; the functions b_{11}, b_{22} vanish on the diagonal, but b_{12} and b_{21} need not. An element $c \in \Omega^2\mathcal{A} \subset \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ can be written as a family of functions $c = \{c_{ijk} \in$

$C^\infty(M \times M \times M) : i,j,k = 1,2\}$, subject to some relations. The algebraic rules for manipulating such expressions can be determined by combining the rules for the commutative and two-point examples. One gets:

$$\begin{aligned}
(ab)_{ij}(x,y) &:= a_i(x) b_{ij}(x,y), \\
(ba)_{ij}(x,y) &:= b_{ij}(x,y) a_j(y), \\
(ac)_{ijk}(x,y,z) &:= a_i(x) c_{ijk}(x,y,z), \\
(ca)_{ijk}(x,y,z) &:= c_{ijk}(x,y,z) a_k(z), \\
(bb')_{ijk}(x,y,z) &:= b_{ij}(x,y) b_{jk}(y,z), \\
(da)_{ij}(x,y) &:= a_j(y) - a_i(x), \\
(db)_{ijk}(x,y,z) &:= b_{jk}(y,z) - b_{ik}(x,z) + b_{ij}(x,y). \\
(b^*)_{ij}(x,y) &:= -b^*_{ji}(y,x), \\
(c^*)_{ijk}(x,y,z) &:= c^*_{kji}(z,y,x).
\end{aligned} \tag{7.4}$$

Writing elements $b \in \Omega^1\mathcal{A}$ in matrix form, we can express the condition $\alpha = f\alpha f$ as

$$\alpha = \begin{pmatrix} b_{1,11} & b_{1,12} & 0 & b_{2,12} \\ b_{1,21} & b_{1,22} & 0 & b_{2,22} \\ 0 & 0 & 0 & 0 \\ b_{3,21} & b_{3,22} & 0 & b_{4,22} \end{pmatrix}, \tag{7.5}$$

where each 2×2 block b_r lies in $\Omega^1\mathcal{A}$. The skew-adjointness of α is now:

$$b^*_{1,ij}(x,y) = b_{1,ji}(y,x), \quad b^*_{2,ij}(x,y) = b_{3,ji}(y,x), \quad b^*_{4,ij}(x,y) = b_{4,ji}(y,x). \tag{7.6}$$

We will also write elements of $\mathcal{A}$ as $a = \sum_r (f^r, f'^r)$ with $f^r, f'^r \in C^\infty(M)$, to minimize the clutter of indices. If $b = (f,f')\, d(g,g')$ is one summand of a typical element of $\Omega^1\mathcal{A}$, we have $\pi(b) \in \mathcal{L}(\mathcal{H})$ given by

$$\begin{aligned}
\pi(b) &= \mathrm{i} \begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix} \left[\begin{pmatrix} \not{\partial} \otimes I & \gamma_5 \otimes m \\ \gamma_5 \otimes m & \not{\partial} \otimes I \end{pmatrix}, \begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix} \right] \\
&= \mathrm{i} \begin{pmatrix} c(f\,\boldsymbol{d}g) \otimes I & -\gamma_5 f \Delta g \otimes m \\ \gamma_5 f' \Delta g \otimes m & c(f'\,\boldsymbol{d}g') \otimes I \end{pmatrix},
\end{aligned} \tag{7.7}$$

where $\Delta g := g - g' \in C^\infty(M)$. Taking finite sums of such operators, we get the reduction rules:

$$\pi(b)_{ij} = \begin{cases} \mathrm{i}c\left(\lim\limits_{y\to x} \dfrac{b_{ij}(x,y)}{x-y}\right) \otimes I & \text{if } i = j, \\ \mathrm{i}\gamma_5\, b_{ij}(x,x) \otimes m & \text{if } i \neq j. \end{cases} \tag{7.8}$$

If we rewrite (7.5) as $\alpha = \sum_r \alpha^r$ (a finite sum), where $\alpha^r(x,y)$ equals

$$\begin{pmatrix} f_1^r(x)(g_1^r(y) - g_1^r(x)) & f_1^r(x)(g_1'^r(y) - g_1^r(x)) & f_2^r(x)(g_2'^r(y) - g_2^r(x)) \\ f_1'^r(x)(g_1^r(y) - g_1'^r(x)) & f_1'^r(x)(g_1'^r(y) - g_1'^r(x)) & f_2'^r(x)(g_2'^r(y) - g_2'^r(x)) \\ f_3'^r(x)(g_3^r(y) - g_3'^r(x)) & f_3'^r(x)(g_3'^r(y) - g_3'^r(x)) & f_4'^r(x)(g_4'^r(y) - g_4'^r(x)) \end{pmatrix}, \tag{7.9}$$

we then get for $\pi(\alpha) \in \mathcal{L}(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H})$:

$$\pi(\alpha) = \mathrm{i} \begin{pmatrix} c(A) \otimes I & \gamma_5(\phi_1^* - 1) \otimes m & \gamma_5 \phi_2^* \otimes m \\ \gamma_5(\phi_1 - 1) \otimes m & c(A') \otimes I & -c(W^*) \otimes I \\ \gamma_5 \phi_2 \otimes m & c(W) \otimes I & c(Z) \otimes I \end{pmatrix}, \tag{7.10}$$

where we have introduced

$$A := \sum_r f_1^r \, \boldsymbol{d} g_1^r, \qquad A' := \sum_r f_1'^r \, \boldsymbol{d} g_1'^r, \qquad Z := \sum_r f_4'^r \, \boldsymbol{d} g_4'^r,$$
$$W := \sum_r f_3'^r \, \boldsymbol{d} g_3'^r = -\Big(\sum_r f_2'^r \, \boldsymbol{d} g_2'^r\Big)^*,$$
$$\phi_1 - 1 := \sum_r f_1'^r \, \Delta g_1^r = -\Big(\sum_r f_1^r \, \Delta g_1^r\Big)^*,$$
$$\phi_2 := \sum_r f_3'^r \, \Delta g_3^r = -\Big(\sum_r f_2^r \, \Delta g_2^r\Big)^*. \tag{7.11}$$

Here A, A' and Z are skew-adjoint one-forms on M, W is a complex one-form, and we have two scalar fields ϕ_1, ϕ_2; we aim to show that these form a Higgs doublet.

7.3. We now compute the Yang–Mills action functional of subsection 6.3. We must first determine $\pi(d\alpha)$. From (7.7) we get for the top left block of $\pi(d\alpha)$:

$$-\sum_r \begin{pmatrix} c(\boldsymbol{d} f_1^r) \otimes I & -\gamma_5 \Delta f_1^r \otimes m \\ \gamma_5 \Delta f_1^r \otimes m & c(\boldsymbol{d} f_1'^r) \otimes I \end{pmatrix} \begin{pmatrix} c(\boldsymbol{d} g_1^r) \otimes I & -\gamma_5 \Delta g_1^r \otimes m \\ \gamma_5 \Delta g_1^r \otimes m & c(\boldsymbol{d} g_1'^r) \otimes I \end{pmatrix}$$
$$= \sum_r \begin{pmatrix} -c(\boldsymbol{d} f_1^r) c(\boldsymbol{d} g_1^r) \otimes I & \gamma_5 c(\Delta f_1^r \, \boldsymbol{d} g_1'^r \\ \quad + \Delta f_1^r \Delta g_1^r \otimes m^2 & \quad - \boldsymbol{d} f_1^r \, \Delta g_1^r) \otimes m \\ \gamma_5 c(\boldsymbol{d} f_1'^r \, \Delta g_1^r & -c(\boldsymbol{d} f_1'^r) c(\boldsymbol{d} g_1'^r) \otimes I \\ \quad - \Delta f_1^r \, \boldsymbol{d} g_1^r) \otimes m & \quad + \Delta f_1^r \Delta g_1^r \otimes m^2 \end{pmatrix}. \tag{7.12}$$

One can simplify:

$$\begin{aligned} &\sum_r \boldsymbol{d} f_1'^r \, \Delta g_1^r - \Delta f_1^r \, \boldsymbol{d} g_1^r \\ &\quad = \sum_r \boldsymbol{d}(f_1'^r \, \Delta g_1^r) + f_1'^r \, \boldsymbol{d} g_1'^r - f_1^r \, \boldsymbol{d} g_1^r \\ &\quad = \boldsymbol{d}\phi_1 + A' - A. \end{aligned} \tag{7.13}$$

Similarly

$$\sum_r \Delta f_1^r \, \boldsymbol{d} g_1'^r - \boldsymbol{d} f_1^r \, \Delta g_1^r = \boldsymbol{d}\phi_1^* + A - A',$$
$$\sum_r \Delta f_1^r \Delta g_1^r = 2 - \phi_1 - \phi_1^*.$$

Moreover,

$$\sum_r c(\boldsymbol{d} f_1^r) c(\boldsymbol{d} g_1^r) = c(\boldsymbol{d} A + \psi_1),$$
$$\sum_r c(\boldsymbol{d} f_1'^r) c(\boldsymbol{d} g_1'^r) = c(\boldsymbol{d} A' + \psi_2),$$

for some scalar fields ψ_1, ψ_2. When one computes the Yang–Mills functional $\mathrm{YM}(\nabla)$ using (6.6), the projector P suppresses the scalar terms $\psi_i \otimes I$, so we can as well assume that each $\psi_i = 0$. Thus the top left block of $\pi(d\alpha)$ is:

$$\begin{pmatrix} -c(\boldsymbol{d}A) \otimes I + (2 - \phi_1 - \phi_1^*) \otimes m^2 & \gamma_5\, c(\boldsymbol{d}\phi_1^* + A - A') \otimes m \\ \gamma_5\, c(\boldsymbol{d}\phi_1 + A' - A) \otimes m & -c(\boldsymbol{d}A') \otimes I + (2 - \phi_1 - \phi_1^*) \otimes m^2 \end{pmatrix}. \tag{7.14}$$

For the third column of $\pi(d\alpha)$, we get similarly:

$$\sum_r \begin{pmatrix} \gamma_5\, c(\Delta f_2^r\, \boldsymbol{d} g_2'^r - \boldsymbol{d} f_2'^r\, \Delta g_2^r) \otimes m \\ -c(\boldsymbol{d} f_2'^r) c(\boldsymbol{d} g_2'^r) \otimes I + \Delta f_2^r \Delta g_2^r \otimes m^2 \\ -c(\boldsymbol{d} f_4'^r) c(\boldsymbol{d} g_4'^r) \otimes I + \Delta f_4^r \Delta g_4^r \otimes m^2 \end{pmatrix}$$
$$= \begin{pmatrix} \gamma_5\, c(\boldsymbol{d}\phi_2^* + W^*) \otimes m \\ c(\boldsymbol{d} W^*) \otimes I - \phi_2^* \otimes m^2 \\ -c(\boldsymbol{d} Z) \otimes I \end{pmatrix}, \tag{7.15}$$

after using (7.11) and suppressing terms of the form $\psi \otimes I$. For the bottom row, we check that

$$\sum_r \boldsymbol{d} f_3'^r\, \Delta g_3^r - \Delta f_3^r\, \boldsymbol{d} g_3^r = \boldsymbol{d}\phi_2 + W,$$
$$\sum_r c(\boldsymbol{d} f_3'^r) c(\boldsymbol{d} g_3'^r) = c(\boldsymbol{d} W + \psi_3),$$

and $\sum_r \Delta f_3^r \Delta g_3^r = -\phi_2$. (Note that sums such as $\sum_r f_3^r \Delta g_3^r$, which do not appear on the right in (7.11), necessarily vanish on account of $\alpha = f \alpha f$.) Putting everything together, we arrive at the following expression for $\pi(d\alpha)$:

$$\begin{pmatrix} -c(\boldsymbol{d}A) & 0 & 0 \\ 0 & -c(\boldsymbol{d}A') & c(\boldsymbol{d}W^*) \\ 0 & -c(\boldsymbol{d}W) & -c(\boldsymbol{d}Z) \end{pmatrix} \otimes I$$
$$+ \begin{pmatrix} 2 - \phi_1 - \phi_1^* & 0 & 0 \\ 0 & 2 - \phi_1 - \phi_1^* & -\phi_2^* \\ 0 & -\phi_2 & 0 \end{pmatrix} \otimes m^2 \tag{7.16}$$
$$+ \begin{pmatrix} 0 & \gamma_5\, c(\boldsymbol{d}\phi_1^* + A - A') & \gamma_5\, c(\boldsymbol{d}\phi_2^* + W^*) \\ \gamma_5\, c(\boldsymbol{d}\phi_1 + A' - A) & 0 & 0 \\ \gamma_5\, c(\boldsymbol{d}\phi_2 + W) & 0 & 0 \end{pmatrix} \otimes m.$$

The curvature θ has image

$$\pi(\theta) = \pi(f\, df\, df) + \pi(f\, d\alpha\, f) + \pi(\alpha)^2 =: R + S, \tag{7.17}$$

where, after subtracting the term S killed by the projector P—each matrix entry S_{ij} is of the form $\psi_{ij} \otimes I$ for some $\psi_{ij} \in C^\infty(M)$—the self-adjoint operator R has components

$$\begin{aligned}
R_{11} &= -c(\boldsymbol{d}A) \otimes I + (1 - |\phi_1|^2 - |\phi_2|^2) \otimes m_\perp^2, \\
R_{21} &= \mathrm{i}\gamma_5\, c(\phi_1(A - A') + \phi_2 W^* - \boldsymbol{d}\phi_1) \otimes m, \\
R_{22} &= -c(\boldsymbol{d}A' - W^* \wedge W) \otimes I + (1 - |\phi_1|^2) \otimes m_\perp^2, \\
R_{31} &= \mathrm{i}\gamma_5\, c(\phi_2(A - Z) - \phi_1 W - \boldsymbol{d}\phi_2) \otimes m, \\
R_{32} &= -c(\boldsymbol{d}W + W \wedge A' + Z \wedge W) \otimes I - \phi_1\phi_2^* \otimes m_\perp^2, \\
R_{33} &= -c(\boldsymbol{d}Z - W \wedge W^*) \otimes I + (1 - |\phi_2|^2) \otimes m_\perp^2.
\end{aligned} \tag{7.18}$$

Note that the subtraction of S will kill matrix multiples of the identity: we have written

$$m_\perp^2 := m^2 - N_G^{-1}\,\mathrm{tr}(m^2), \tag{7.19}$$

to denote the orthogonal projection of m—in the Hilbert–Schmidt space of matrices—on the orthogonal complement of the multiples of the identity (see the discussion at the end of this subsection).

The Yang–Mills functional can therefore be expressed as a sum of three terms:

$$\mathrm{YM}(\nabla) = I_2 + I_1 + I_0, \tag{7.20}$$

where I_k is of the form $\int_M \|\eta_k\|^2$ by (5.19), with $\eta_k \in \mathcal{E}^k(M)$. Omitting the common multiplicative constant $1/8\pi^2$ (which amounts to normalizing the Yang–Mills functional), we find explicitly:

$$\begin{aligned}
I_2 &= N_G \int_M (\|\boldsymbol{d}A\|^2 + \|\boldsymbol{d}A' - W^* \wedge W\|^2 \\
&\qquad + \|\boldsymbol{d}Z - W \wedge W^*\|^2 + 2\|\boldsymbol{d}W + W \wedge A' + Z \wedge W\|^2), \\
I_1 &= 2\,\mathrm{tr}(m^2) \int_M (\|\boldsymbol{d}\phi_1 + (A' - A)\phi_1 - W^*\phi_2\|^2 \\
&\qquad + \|\boldsymbol{d}\phi_2 + W\phi_1 + (Z - A)\phi_2\|^2), \\
I_0 &= \mathrm{tr}((m_\perp^2)^2) \int_M 1 + 2(1 - |\phi_1|^2 - |\phi_2|^2)^2.
\end{aligned} \tag{7.21}$$

It is clear that I_2 is the pure gauge part of the Lagrangian; indeed, if we introduce the covariant derivative

$$\boldsymbol{D} := \begin{pmatrix} \boldsymbol{d} + A' & -W^* \\ W & \boldsymbol{d} + Z \end{pmatrix}, \tag{7.22}$$

we have $I_2 = N_G \int_M \mathcal{L}_2$, with

$$\mathcal{L}_2 = \|\boldsymbol{d}A\|^2 + \left\| \boldsymbol{D} \begin{pmatrix} A' & -W^* \\ W & Z \end{pmatrix} \right\|^2. \tag{7.23}$$

It is also clear now that the pair of scalar fields $\Phi = \binom{\phi_1}{\phi_2}$, arising from the cross-terms between the two leaves of our spacetime, is to be interpreted as a doublet of Higgs bosons. Indeed, I_1 may be rewritten as

$$I_1 = 2\,\mathrm{tr}(m^2) \int_M \|(\boldsymbol{D} - A)\Phi\|^2, \tag{7.24}$$

representing the kinetic term for the Higgs fields; and I_0 is then the Higgs self-interaction term. Notice that its form is almost identical to the action (6.23) for the two-point example; that example should thus be thought of as a "pure Higgs" construction, which thereby finds its natural home in noncommutative geometry.

Notice also that the Higgs self-interaction term I_0 is proportional to $\mathrm{tr}((m_\perp^2)^2)$, where $m_\perp^2$ is traceless, on account of the projection involved in computing the Yang–Mills functional (6.6). This reflects the difference between $\pi(\Omega^2)$ and Ω_D^2, mathematically speaking, and has also a transparent physical interpretation: whereas in the usual version of the Glashow–Weinberg–Salam model the existence of the Higgs potential has nothing to do with the number of generations, here, if all the electron-like fermions in the various generations had the same mass, *in particular, if there were only one generation, we would not have a Higgs potential.*

7.4. Thus far, we have obtained essentially the boson part of the GWS model. The main point of noncommutative geometry has been made and rewriting that part in Minkowskian form is trivial. For a more precise identification of the various terms, we refer to the treatment of the full Standard Model in the next section.

The computation of the fermionic action is more of an afterthought in noncommutative geometry. We first "Wick-rotate" $I_F(\psi) = \langle \psi \mid D\psi \rangle - \mathrm{i}\langle \psi \mid \pi(\alpha)\psi \rangle$ and then impose $(\gamma_5 \otimes \sigma_3)\psi = \psi$. The first of these terms gives the integral over M of

$$\mathrm{i}(\bar e_{\mathrm R} \partial\!\!\!/ e_{\mathrm R} + \bar e_{\mathrm L} \partial\!\!\!/ e_{\mathrm L} + \bar\nu_{\mathrm L} \partial\!\!\!/ \nu_{\mathrm L}) - (\bar e_{\mathrm L} m e_{\mathrm R} + \bar e_{\mathrm R} m e_{\mathrm L}). \tag{7.25}$$

Employing (7.10), the second term is the integral of

$$\begin{aligned} &\mathrm{i}(\bar e_{\mathrm R} A\!\!\!/ e_{\mathrm R} + \bar e_{\mathrm L} A\!\!\!/' e_{\mathrm L} + \bar\nu_{\mathrm L} W\!\!\!\!/ e_{\mathrm L} - \bar e_{\mathrm L} W\!\!\!\!/^* \nu_{\mathrm L} + \bar\nu_{\mathrm L} Z\!\!\!/ \nu_{\mathrm L}) \\ &\quad - \bar e_{\mathrm L}(\phi_1 - 1) m e_{\mathrm R} - \bar e_{\mathrm R}(\phi_1^* - 1) m e_{\mathrm L} - \bar\nu_{\mathrm L}\phi_2 m e_{\mathrm R} - \bar e_{\mathrm R}\phi_2^* m \nu_{\mathrm L}, \end{aligned} \tag{7.26}$$

where we have written $A\!\!\!/ = \gamma^\mu A_\mu$ instead of $c(A)$. Adding these expressions together, we get

$$I_F(\psi) = J_1 + J_0, \tag{7.27}$$

with

$$J_1 = \mathrm{i}\int_M (\bar{e}_R(\partial\!\!\!/ + A\!\!\!/)e_R + \bar{e}_L(\partial\!\!\!/ + A\!\!\!/')e_L$$

$$+ \bar{\nu}_L(\partial\!\!\!/ + Z\!\!\!/)\nu_L + \bar{\nu}_L W\!\!\!\!/\, e_L - \bar{e}_L W\!\!\!\!/^{*}\nu_L),$$

$$J_0 = -\int_M (\bar{e}_L\phi_1 m e_R + \bar{e}_R\phi_1^* m e_L + \bar{\nu}_L\phi_2 m e_R + \bar{e}_R\phi_2^* m\nu_L). \tag{7.28}$$

The full action functional $\mathrm{YM}(\nabla) + I_F(\psi) = I_2 + I_1 + I_0 + J_1 + J_0$ thus contains five terms corresponding to those of the GWS model. The integrand of J_1 is simply the fermion kinetic term of the Lagrangian with a minimal coupling to the gauge field; recall that A, A', Z are skew-adjoint fields. The J_0 term gives the Yukawa coupling of the fermions of left and right chirality; recall that m is a positive matrix, whose eigenvalues are the various lepton masses.

7.5. The local gauge group for this model is $\mathrm{U}(1)\times\mathrm{U}(2)$, rather than $\mathrm{U}(1)\times\mathrm{SU}(2)$. We identify the latter group with the subgroup

$$G := \{\,(v,u)\in\mathrm{U}(1)\times\mathrm{U}(2) : v = \det u\,\}, \tag{7.29}$$

via the isomorphism $(v,u)\mapsto(v,v^{-1}u)$. At the Lie algebra level, the condition $v = \det u$ gives:

$$A = A' + Z. \tag{7.30}$$

The somewhat ad-hoc replacement $A'\mapsto A-Z$ gives a quick recipe which reduces the action computed here to the expected form of the GWS model. In the next section we will reexamine the matter, to obtain a more meaningful procedure for the reduction of the gauge group when a quark sector is present.

8. The full Standard Model

8.1. To reflect on the construction of the Standard Model in noncommutative geometry, we begin with the gauge group. An isodoublet of quarks, such as $\binom{d_L}{u_L}$, comes in three colors

$$\begin{pmatrix} d_L^r & d_L^y & d_L^b \\ u_L^r & u_L^y & u_L^b \end{pmatrix},$$

which is acted on by the representation $\mathbf{2}\otimes\mathbf{3}$ of $\mathrm{SU}(2)\times\mathrm{SU}(3)$. So far we have worked under the assumption that the full gauge group could be represented as $\mathcal{U}_\mathcal{A} = \{\,u\in\mathcal{A} : u^*u = uu^* = 1\,\}$ for some involutive algebra $\mathcal{A}$ (perhaps with an added unimodularity condition), and that the representation of the gauge group on the Hilbert space of fermions could be obtained by restriction from that of $\mathcal{A}$. This is certainly true for a $\mathrm{U}(N)$ model (or an $\mathrm{SU}(N)$ model), but the representation $\mathbf{2} \otimes \mathbf{3}$ cannot be so obtained, because $*$-algebra representations cannot in general be tensored. We therefore need two separate algebras $\mathcal{A}$ and $\mathcal{B}$, incorporating electroweak and color gauge symmetries, respectively, with commuting actions on the underlying Hilbert space; the gauge group is then a unimodular subgroup of $\mathcal{U}_{\mathcal{A}} \times \mathcal{U}_{\mathcal{B}}$. This double algebra action is suggested in noncommutative geometry by the Poincaré duality mapping.

8.2. We will keep to the scheme of using an algebra of type $C^\infty(M) \otimes \mathcal{A}_F$ where $\mathcal{A}_F$ is a finite-dimensional algebra. We examine the finite part first. In the GWS model, we let $\mathcal{A}_F$ be the algebra $\mathbb{C} \oplus \mathbb{C}$ acting on a vector bundle over the two-point space $\{q_1, q_2\}$ with fibers $E_{q_1} = \mathbb{C}$, $E_{q_2} = \mathbb{C}^2$. Alternatively, we could replace $\mathcal{A}_F$ by $\mathbb{C} \oplus \mathbb{C}^{2\times 2}$ acting on $E = \mathbb{C} \oplus \mathbb{C}^2$ in the obvious way; nothing is lost by dropping the commutativity (which is the whole point of noncommutative geometry); and the formulae for connections and curvatures simplify, since one may replace $pd + \alpha$ by $d + \alpha$.

However, the presence of the conjugate Higgs field $\tilde{\Phi} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\Phi^* =: J\Phi^*$ in the Yukawa term, which is necessary to give mass to *both* quarks (whereas in the leptonic sector the neutrino remains massless), suggests that the SU(2) gauge group be introduced directly by restricting the $\mathbb{C}^{2\times 2}$ summand of $\mathcal{A}$ by the condition $JxJ^{-1} = x^*$. For $\{u \in \mathrm{U}(2) : JuJ^{-1} = u^*\} = \mathrm{SU}(2)$ this realizes the unitary equivalence between the fundamental representation of SU(2) and its contragredient representation. Of course, $\{x \in \mathbb{C}^{2\times 2} : JxJ^{-1} = x^*\}$ gives precisely the *quaternion* algebra $\mathbb{H}$, so the best choice of $\mathcal{A}_F$ is the *real* algebra $\mathbb{C} \oplus \mathbb{H}$ rather than $\mathbb{C} \oplus \mathbb{C}^{2\times 2}$.

If $q \in \mathbb{H}$, we can write $q = \alpha + \beta j = \alpha + j\beta^*$ with $\alpha, \beta \in \mathbb{C}$; the corresponding element of $\mathbb{C}^{2\times 2}$ is $\left(\begin{smallmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{smallmatrix}\right)$. Let us recall that quaternion multiplication is given by $(\alpha + \beta \mathrm{j})(\sigma + \tau \mathrm{j}) = (\alpha\sigma - \beta\tau^*) + (\alpha\tau + \beta\sigma^*)\mathrm{j}$.

The color symmetry has gauge group SU(3), acting trivially on leptons and by its fundamental representation $\mathbf{3}$ on quarks; so the corresponding algebra is naturally $\mathbb{C} \oplus \mathbb{C}^{3\times 3}$.

8.3. The finite-space part of the proposed model is therefore a graded $\mathcal{A} \otimes \mathcal{B}$ module $(\mathfrak{h}_F, D_F, \gamma_F)$ with $\mathcal{A} = \mathbb{C} \oplus \mathbb{H}$, $\mathcal{B} = \mathbb{C} \oplus \mathbb{C}^{3\times 3}$. We may decompose $\mathfrak{h}_F$ as follows:

$$\mathfrak{h}_F = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_1 = \mathfrak{h}_0 \oplus (\mathfrak{h}_1 \otimes \mathbb{C}^3), \tag{8.1}$$

with $\mathcal{B}$ acting by scalars on $\mathfrak{h}_0$, $\mathfrak{h}_1$, each of which carries a $*$-representation of $\mathcal{A}$.

A finite-dimensional $*$-representation of the real involutive algebra $\mathcal{A}$ is of the general form

$$\pi(\lambda, q) = \lambda I_{n^+} \oplus \lambda^* I_{n^-} \oplus \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \otimes I_m.$$

Since there are no right-handed neutrinos, the lepton sector $\mathfrak{h}_0$ has dimension $3N_{\mathrm{G}}$, and the quark sector $\mathfrak{h}_1$ has dimension $4N_{\mathrm{G}}$, where N_{G} is the number of generations. For a typical vector, we use the suggestive notation:

$$\psi_0 := \begin{pmatrix} e_{\rm R} \\ e_{\rm L} \\ \nu_{\rm L} \end{pmatrix}, \qquad \psi_1 := \begin{pmatrix} d_{\rm R} \\ u_{\rm R} \\ d_{\rm L} \\ u_{\rm L} \end{pmatrix}, \tag{8.2}$$

where each $e_{\rm R}$, $d_{\rm R}$, etc. denotes an $N_{\rm G}$-tuple with one representative in each generation. Accordingly, we choose the representations π_0, π_1 of $\mathcal{A}$ on $\mathfrak{h}_0$, $\mathfrak{h}_1$ respectively as

$$\pi_0(\lambda, q) := \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta^* & \alpha^* \end{pmatrix} \otimes I_{N_{\rm G}},$$

$$\pi_1(\lambda, q) := \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^* & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta^* & \alpha^* \end{pmatrix} \otimes I_{N_{\rm G}}. \tag{8.3}$$

In summary, $\mathfrak{h}_F = [\mathbb{C} \oplus \mathbb{H} \oplus (\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{H}) \otimes \mathbb{C}^3] \otimes \mathbb{R}^{N_{\rm G}}$. Note that $\pi := \pi_0 \oplus (\pi_1 \otimes I_3)$ is faithful. The grading operator is given by $\gamma_F = \pi(1, -1)$; since $(1, -1)$ lies in the centre of $\mathcal{A}$, both $\mathcal{A}$ and $\mathcal{B}$ act by even operators on $\mathfrak{h}_F$.

We choose the self-adjoint odd operator $D_F := D_0 \oplus D_1 \otimes I_3$ as

$$D_0 := \begin{pmatrix} 0 & m_{\rm e} & 0 \\ m_{\rm e} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad D_1 := \begin{pmatrix} 0 & 0 & m_{\rm d}^\dagger & 0 \\ 0 & 0 & 0 & m_{\rm u} \\ m_{\rm d} & 0 & 0 & 0 \\ 0 & m_{\rm u} & 0 & 0 \end{pmatrix}. \tag{8.4}$$

Here $m_{\rm e}$, $m_{\rm u}$ are *real* positive-definite $N_{\rm G} \times N_{\rm G}$ matrices. Also $m_{\rm d}$ is positive-definite, but we cannot assume that it is diagonalized simultaneously with $m_{\rm u}$. The matrix that will interchange the orthonormal basis given by the mass eigenstates of the u-quarks and the d-quarks is essentially the Kobayashi–Maskawa matrix. Note that D_F commutes with $\mathcal{B}$, so that the condition $[[D_F, \pi(a)], b] = 0$ is satisfied, and it commutes also with the diagonal subalgebra $\{ \pi(\lambda, \lambda) : \lambda \in \mathbb{C} \}$ of $\pi(\mathcal{A})$.

Notice also that if one suppresses the second row and column of $\pi_1(\lambda, q)$ and of D_1, these reduce to copies of $\pi_0(\lambda, q)$ and D_0, so we need only compute with π_1 when exploring further.

8.4. We must identify the $\mathcal{A}$-bimodules $\Omega_D^1(\mathcal{A})$ and $\Omega_D^2(\mathcal{A})$. Now the π-homomorphism is determined by the action of $\mathcal{A}$ by π_1 on the K-cycle $(\mathfrak{h}_1, D_1)$. It is straightforward to check that

$$-\mathrm{i}\pi_1\left(\textstyle\sum_r a_0^r \, da_1^r\right) = \sum_r \pi_1(\lambda_0^r, q_0^r)\, [D_1, \pi_1(\lambda_1^r, q_1^r)] = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}, \tag{8.5}$$

with

$$X = \begin{pmatrix} m_\mathrm{d}^\dagger & 0 \\ 0 & m_\mathrm{u} \end{pmatrix} \begin{pmatrix} \alpha_{12} & \beta_{12} \\ -\beta_{12}^* & \alpha_{12}^* \end{pmatrix} = \boldsymbol{m}^\dagger q_{12},$$
$$Y = \begin{pmatrix} \alpha_{21} & \beta_{21} \\ -\beta_{21}^* & \alpha_{21}^* \end{pmatrix} \begin{pmatrix} m_\mathrm{d} & 0 \\ 0 & m_\mathrm{u} \end{pmatrix} = q_{21}\boldsymbol{m}, \tag{8.6}$$

where we have abbreviated

$$\boldsymbol{m} := \begin{pmatrix} m_\mathrm{d} & 0 \\ 0 & m_\mathrm{u} \end{pmatrix}, \tag{8.7}$$

and

$$\alpha_{12} = \sum_r \lambda_0^r(\alpha_1^r - \lambda_1^r), \qquad \beta_{12} = \sum_r \lambda_0^r \beta_1^r,$$
$$\alpha_{21} = \sum_r \alpha_0^r(\lambda_1^r - \alpha_1^r) + \beta_0^r\beta_1^{r*}, \qquad \beta_{21} = \sum_r \beta_0^r(\lambda_1^{r*} - \alpha_1^{r*}) - \alpha_0^r\beta_1^r;$$

thus

$$q_{12} := \alpha_{12} + \beta_{12} j = \sum_r \lambda_0^r(q_1^r - \lambda_1^r),$$
$$q_{21} := \alpha_{21} + \beta_{21} j = \sum_r q_0^r(\lambda_1^r - q_1^r), \tag{8.8}$$

and so we have

$$\pi_1(\textstyle\sum_r a_0^r\, da_1^r) = \mathrm{i}\begin{pmatrix} 0 & \boldsymbol{m}^\dagger q_{12} \\ q_{21}\boldsymbol{m} & 0 \end{pmatrix}, \tag{8.9}$$

analogously to (5.24). Thus $\Omega_D^1(\mathcal{A}) = \mathbb{H} \oplus \mathbb{H}$.

Before plunging into the determination of $\Omega_D^2(\mathcal{A})$, notice that $\boldsymbol{m}\, q = q\boldsymbol{m}$ iff q is complex. Indeed, if $q = \alpha + \beta\mathrm{j}$ with α, β complex, then

$$[\boldsymbol{m}, q] = (m_\mathrm{d} - m_\mathrm{u}) \otimes \begin{pmatrix} 0 & \beta \\ \beta^* & 0 \end{pmatrix},$$
$$[\boldsymbol{m}\boldsymbol{m}^\dagger, q] = (m_\mathrm{d} m_\mathrm{d}^\dagger - m_\mathrm{u}^2) \otimes \begin{pmatrix} 0 & \beta \\ \beta^* & 0 \end{pmatrix}.$$

From (8.9),

$$\pi_1(q_{12}, q_{21})\, \pi_1(q_{12}', q_{21}') = -\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}, \tag{8.10}$$

where

$$Z = \boldsymbol{m}^\dagger q_{12} q_{21}' \boldsymbol{m}, \qquad W = q_{21}\boldsymbol{m}\boldsymbol{m}^\dagger q_{12}'. \tag{8.11}$$

On the other hand, we have

$$\pi_1(\textstyle\sum_r da_0^r\, da_1^r) = -\begin{pmatrix} \boldsymbol{m}^\dagger(q_{12} + q_{21})\boldsymbol{m} & 0 \\ 0 & \boldsymbol{m}\boldsymbol{m}^\dagger q_{12} + q_{21}\boldsymbol{m}\boldsymbol{m}^\dagger + X \end{pmatrix}, \tag{8.12}$$

where

$$X = \sum_r q_0^r \, [\boldsymbol{m}\boldsymbol{m}^\dagger, q_1^r]$$
$$= (m_\mathrm{d} m_\mathrm{d}^\dagger - m_\mathrm{u}^2) \otimes \sum_r \begin{pmatrix} \beta_0^r \beta_1^{r*} & \alpha_0^r \beta_1^r \\ \alpha_0^{r*} \beta_1^{r*} & -\beta_0^{r*} \beta_1^r \end{pmatrix}. \tag{8.13}$$

Note that X is not a quaternionic block matrix, but is rather of the form $(m_\mathrm{d} m_\mathrm{d}^\dagger - m_\mathrm{u}^2) \otimes \left(\begin{smallmatrix} \lambda & \mu \\ \mu^* & -\lambda^* \end{smallmatrix}\right)$ with λ, μ complex numbers. If V denotes the vector space of such elements, then by examining $\pi_1(d\alpha)$ when $\pi_1(\alpha) = 0$ we conclude at once $V = \pi_1(dJ_0^1)$. Now it is clear that $\Omega_D^2(\mathcal{A}) = \mathbb{H} \oplus \mathbb{H}$ and that $d: \Omega_D^1(\mathcal{A}) \to \Omega_D^2(\mathcal{A})$ is given by $d(q_{12}, q_{21}) = (q_{12} + q_{21}, q_{12} + q_{21})$.

Let $\alpha \in \Omega_D^1(\mathcal{A})$ be skew-adjoint. Since $(q_{12}, q_{21})^* = (-q_{21}^*, -q_{12}^*)$, this means that $\alpha = (q, q^*)$. The vector bundle is $\mathcal{A}$ itself; then the curvature of the connection $\pi_D(d + \alpha)$ is

$$\theta = \pi_D(d\alpha + \alpha^2) = (q^* + q + qq^*, q + q^* + q^*q)$$
$$= (|1 + q|^2 - 1, |1 + q|^2 - 1). \tag{8.14}$$

Thus

$$\mathrm{Tr}(\theta^2) = [\tfrac{3}{2}(\mathrm{tr}\, m_\mathrm{u}^2)^2 + \tfrac{3}{2}(\mathrm{tr}\, |m_\mathrm{d}|^2)^2 + \mathrm{tr}\, m_\mathrm{u}^2 |m_\mathrm{d}|^2] (|1 + q|^2 - 1).$$

The minimal connections form a three-sphere: $1 + q \in \mathrm{SU}(2)$, just as for the finite-space part of the GWS model.

8.5. With the necessary bookkeeping for the finite part now completed, we turn to the full model. We take a compact four-dimensional spin^c manifold, with K-cycle $(\Gamma^2(S), \not{\partial}, \gamma_5)$, of the Dirac type and construct the following K-cycle $(\mathcal{H}, D, \Gamma)$ as a module over the algebra $\mathcal{A} \otimes \mathcal{B}$, where now

$$\mathcal{A} := C^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{H}), \qquad \mathcal{B} := C^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{C}^{3\times 3}). \tag{8.15}$$

We have $\mathcal{H} = \Gamma^2(S) \otimes (\mathfrak{h}_0 \oplus (\mathfrak{h}_1 \otimes \mathbb{C}^3)) =: \mathcal{H}_0 \oplus (\mathcal{H}_1 \otimes \mathbb{C}^3)$, where $\Gamma^2(S)$ is the Hilbert space of square-integrable sections of the spinor bundle over M, $\mathfrak{h}_0 = \mathbb{C}^{3N_\mathrm{G}}$ and $\mathfrak{h}_1 = \mathbb{C}^{4N_\mathrm{G}}$ as before; $\mathcal{B}$ acts on $\mathcal{H}$ in the obvious manner and $\mathcal{A}$ acts on $\mathcal{H}_0$ and on $\mathcal{H}_1$ by representations π_0, π_1 extending those of (8.3). The grading operator is $\Gamma = \gamma_5 \otimes \gamma_F$, and $D = \not{\partial} \otimes I + \gamma_5 \otimes D_F$, i.e., $D = D_0 + (D_1 \otimes I_3)$, where now

$$D_0 = \begin{pmatrix} \not{\partial} \otimes I & \gamma_5 \otimes m_\mathrm{e} & 0 \\ \gamma_5 \otimes m_\mathrm{e} & \not{\partial} \otimes I & 0 \\ 0 & 0 & \not{\partial} \otimes I \end{pmatrix},$$
$$D_1 = \begin{pmatrix} \not{\partial} \otimes I & 0 & \gamma_5 \otimes m_\mathrm{d}^\dagger & 0 \\ 0 & \not{\partial} \otimes I & 0 & \gamma_5 \otimes m_\mathrm{u} \\ \gamma_5 \otimes m_\mathrm{d} & 0 & \not{\partial} \otimes I & 0 \\ 0 & \gamma_5 \otimes m_\mathrm{u} & 0 & \not{\partial} \otimes I \end{pmatrix}. \tag{8.16}$$

The bimodule $\Omega^1_D(\mathcal{A})$ is readily identified. Given $a_0^r = (f_0^r, q_0^r), a_1^r = (f_1^r, q_1^r)$, where $f_i^r \in C^\infty(M,\mathbb{C})$ and $q_i^r \in C^\infty(M,\mathbb{H})$ are complex- and quaternion-valued functions, respectively, on M, let

$$\alpha := \sum_r a_0^r \, da_1^r \in \Omega^1\mathcal{A}. \tag{8.17}$$

We now find that

$$\pi_1(\alpha) = \mathrm{i}\sum_r \pi_1(f_0^r, q_0^r)\,[D_1, \pi_1(f_1^r, q_1^r)] = \mathrm{i}\begin{pmatrix} P & X \\ Y & Q \end{pmatrix}, \tag{8.18}$$

with $X = \gamma_5 \boldsymbol{m}^\dagger q_{12}$, $Y = \gamma_5 q_{21}\boldsymbol{m}$, and

$$P = \begin{pmatrix} c(A)\otimes I & 0 \\ 0 & c(A^*)\otimes I \end{pmatrix}, \qquad Q = \begin{pmatrix} c(W_1)\otimes I & c(W_2)\otimes I \\ -c(W_2^*)\otimes I & c(W_1^*)\otimes I \end{pmatrix}, \tag{8.19}$$

with

$$q_{12} = \sum_r f_0^r(q_1^r - f_1^r), \qquad q_{21} = \sum_r q_0^r(f_1^r - q_1^r),$$
$$A = \sum_r f_0^r\, df_1^r, \qquad W = W_1 + W_2\mathrm{j} = \sum_r q_0^r\, dq_1^r. \tag{8.20}$$

We see that q_{12} and q_{21} lie in $C^\infty(M,\mathbb{H})$, and that A and W are ordinary one-forms on M, $\mathbb{C}$-valued and $\mathbb{H}$-valued, respectively. Therefore

$$\Omega^1_D(\mathcal{A}) = \mathcal{E}^1(M,\mathbb{C}) \oplus C^\infty(M,\mathbb{H}) \oplus C^\infty(M,\mathbb{H}) \oplus \mathcal{E}^1(M,\mathbb{H}); \tag{8.21}$$

a typical element may be denoted (A, q_{12}, q_{21}, W). The algebraic rules for $\Omega^1_D(\mathcal{A})$ are

$$\begin{aligned}
(f,q)\,(A,q_{12},q_{21},W) &= (fA, fq_{12}, qq_{21}, qW), \\
(A,q_{12},q_{21},W)\,(f,q) &= (Af, q_{12}q, q_{21}f, Wq), \\
(A,q_{12},q_{21},W)^* &= (A^*, -q_{21}^*, -q_{12}^*, W^*), \\
d(f,q) &= (\boldsymbol{d}f, q-f, f-q, \boldsymbol{d}q).
\end{aligned} \tag{8.22}$$

If $\alpha \in \Omega^1\mathcal{A}$ is a skew-adjoint one-form, then we have $A^* = -A$, $W^* = -W$, and $q_{12}^* = q_{21}$.

8.6. We rewrite (8.18) in block matrix form as

$$\pi_1(\alpha) = \mathrm{i}\begin{pmatrix} c(A)\otimes I & \gamma_5 \boldsymbol{m}^\dagger q_{12} \\ \gamma_5 q_{21}\boldsymbol{m} & c(W)\otimes I \end{pmatrix}. \tag{8.23}$$

From this we find

$$-\pi_1(\alpha^2) = \begin{pmatrix} c(A^2)\otimes I + \boldsymbol{m}^\dagger q_{12}q_{21}\boldsymbol{m} & \gamma_5 \boldsymbol{m}^\dagger c(q_{12}W - Aq_{12}) \\ \gamma_5\, c(q_{21}A - Wq_{21})\boldsymbol{m} & c(W^2)\otimes I + q_{21}\boldsymbol{m}\boldsymbol{m}^\dagger q_{12} \end{pmatrix}. \tag{8.24}$$

Moreover, since $d\alpha = \sum_r d(f_0^r, q_0^r)\, d(f_1^r, q_1^r) \in \Omega^2\mathcal{A}$, we have

$$-\pi_1(d\alpha) = \sum_r \begin{pmatrix} c(\boldsymbol{d}f_0^r)\otimes I & \gamma_5 \boldsymbol{m}^\dagger(q_0^r - f_0^r) \\ \gamma_5(f_0^r - q_0^r)\boldsymbol{m} & c(\boldsymbol{d}q_0^r)\otimes I \end{pmatrix} \times \begin{pmatrix} c(\boldsymbol{d}f_1^r)\otimes I & \gamma_5 \boldsymbol{m}^\dagger(q_1^r - f_1^r) \\ \gamma_5(f_1^r - q_1^r)\boldsymbol{m} & c(\boldsymbol{d}q_1^r)\otimes I \end{pmatrix}. \tag{8.25}$$

The entries of the product block matrix may be simplified as follows, using the technique developed in section 7:

$$\begin{aligned} -\pi_1(d\alpha)_{11} &= \textstyle\sum_r c(\boldsymbol{d}f_0^r\,\boldsymbol{d}f_1^r)\otimes I + \boldsymbol{m}^\dagger(q_0^r - f_0^r)(f_1^r - q_1^r)\boldsymbol{m} \\ &= c(\boldsymbol{d}A + \psi)\otimes I + \boldsymbol{m}^\dagger(q_{12} + q_{21})\boldsymbol{m}, \\ -\pi_1(d\alpha)_{12} &= \gamma_5 \boldsymbol{m}^\dagger c\big(\textstyle\sum_r -\boldsymbol{d}f_0^r(q_1^r - f_1^r) + (q_0^r - f_0^r)\boldsymbol{d}q_1^r\big) \\ &= -\gamma_5 \boldsymbol{m}^\dagger c(\boldsymbol{d}q_{12} + A - W), \\ -\pi_1(d\alpha)_{21} &= \gamma_5\, c\big(\textstyle\sum_r (f_0^r - q_0^r)\boldsymbol{d}f_1^r) - \boldsymbol{d}q_0^r(f_1^r - q_1^r)\big)\,\boldsymbol{m} \\ &= -\gamma_5\, c(\boldsymbol{d}q_{21} - A + W)\,\boldsymbol{m}, \\ -\pi_1(d\alpha)_{22} &= \textstyle\sum_r c(\boldsymbol{d}q_0^r\,\boldsymbol{d}q_1^r)\otimes I + (f_0^r - q_0^r)\boldsymbol{m}\boldsymbol{m}^\dagger(q_1^r - f_1^r) \\ &= c(\boldsymbol{d}W + \chi)\otimes I + \boldsymbol{m}\boldsymbol{m}^\dagger q_{12} + q_{21}\boldsymbol{m}\boldsymbol{m}^\dagger + X, \end{aligned} \tag{8.26}$$

where ψ, χ are complex- and quaternionic-valued functions, respectively. Let V denote now the vector space with elements of the form $(m_d m_d^\dagger - m_u^2)\otimes\left(\begin{smallmatrix}\lambda & \mu \\ \mu^* & -\lambda^*\end{smallmatrix}\right)$ with $\lambda, \mu \in C^\infty(M,\mathbb{C})$. By examining $\pi_1(d\alpha)$ when $\pi_1(\alpha) = 0$ (i.e., when $A = 0$, $W = 0$, $q_{12} = q_{21} = 0$), it yields

$$S \in \pi_1(dJ_0^1) \iff S = \begin{pmatrix} \psi\otimes I & 0 \\ 0 & \chi\otimes I + X \end{pmatrix}, \tag{8.27}$$

where $\psi \in C^\infty(M,\mathbb{C})$, $\chi \in C^\infty(M,\mathbb{H})$, and $X \in V$.

The image of the curvature $\theta = d\alpha + \alpha^2$ may now be obtained. Let us write $\phi_{12} := q_{12} + 1$, $\phi_{21} := q_{21} + 1$. Adding (8.24) and (8.25), we get an expression for $\pi_1(\theta)$:

$$\begin{aligned} -\pi_1(\theta)_{11} &= c(\boldsymbol{d}A + \psi')\otimes I + (|\phi_{12}|^2 - 1)\boldsymbol{m}^\dagger\boldsymbol{m}, \\ -\pi_1(\theta)_{12} &= -\gamma_5 \boldsymbol{m}^\dagger c(\boldsymbol{d}\phi_{12} + A\phi_{12} - \phi_{12}W), \\ -\pi_1(\theta)_{21} &= -\gamma_5\, c(\boldsymbol{d}\phi_{21} - \phi_{21}A + W\phi_{21})\boldsymbol{m}, \\ -\pi_1(\theta)_{22} &= c(\boldsymbol{d}W + W\wedge W + \chi')\otimes I + \phi_{21}\boldsymbol{m}\boldsymbol{m}^\dagger\phi_{12} - \boldsymbol{m}\boldsymbol{m}^\dagger + X. \end{aligned} \tag{8.28}$$

If $\phi_{12} = \alpha_{12} + \beta_{12}\mathrm{j}$, $\phi_{21} = \alpha_{21} + \beta_{21}\mathrm{j}$, then

$$\begin{aligned} &\phi_{21}\boldsymbol{m}\boldsymbol{m}^\dagger\phi_{12} - \boldsymbol{m}\boldsymbol{m}^\dagger \\ &= \begin{pmatrix} (\alpha_{21}\alpha_{12} - 1)m_d m_d^\dagger - \beta_{21}\beta_{12}^* m_u^2 & \alpha_{21}\beta_{12} m_d m_d^\dagger + \beta_{21}\alpha_{12}^* m_u^2 \\ -\beta_{21}^*\alpha_{12} m_d m_d^\dagger - \alpha_{21}^*\beta_{12}^* m_u^2 & -\beta_{21}^*\beta_{12} m_d m_d^\dagger + (\alpha_{21}^*\alpha_{12}^* - 1)m_u^2 \end{pmatrix} \\ &= \tfrac12(m_d m_d^\dagger + m_u^2)(\phi_{21}\phi_{12} - 1) + \tfrac12(m_d m_d^\dagger - m_u^2)\begin{pmatrix} \lambda & \mu \\ \mu^* & -\lambda^* \end{pmatrix} \end{aligned} \tag{8.29}$$

with $\lambda = \alpha_{21}\alpha_{12} + \beta_{21}\beta_{12}^* - 1$, $\mu = \alpha_{21}\beta_{12} - \beta_{21}\alpha_{12}^*$. The second term on the right hand side lies in V, and hence in $\pi_1(dJ_0^1)$. Finally, we get $\pi_1(\theta) = R_1 + S_1$, where S_1 is the component in $\pi_1(dJ_0^1)$, and

$$\begin{aligned}
(R_1)_{11} &= -c(\boldsymbol{d}A) \otimes I + (1 - |\phi_{12}|^2)(\boldsymbol{m}^\dagger \boldsymbol{m})_\perp, \\
(R_1)_{12} &= \gamma_5 \boldsymbol{m}^\dagger c(\boldsymbol{d}\phi_{12} + A\phi_{12} - \phi_{12}W), \\
(R_1)_{21} &= \gamma_5 c(\boldsymbol{d}\phi_{21} - \phi_{21}A + W\phi_{21})\boldsymbol{m}, \\
(R_1)_{22} &= -c(\boldsymbol{d}W + W \wedge W) \otimes I + \tfrac{1}{2}(1 - |\phi_{12}|^2)(m_\mathrm{d} m_\mathrm{d}^\dagger + m_\mathrm{u}^2)_\perp.
\end{aligned} \tag{8.30}$$

Here

$$\begin{aligned}
(\boldsymbol{m}^\dagger \boldsymbol{m})_\perp &= \boldsymbol{m}^\dagger \boldsymbol{m} - \tfrac{1}{2} N_\mathrm{G}^{-1} \operatorname{tr}(m_\mathrm{d} m_\mathrm{d}^\dagger + m_\mathrm{u}^2), \\
(m_\mathrm{d} m_\mathrm{d}^\dagger + m_\mathrm{u}^2)_\perp &= m_\mathrm{d} m_\mathrm{d}^\dagger + m_\mathrm{u}^2 - N_\mathrm{G}^{-1} \operatorname{tr}(m_\mathrm{d} m_\mathrm{d}^\dagger + m_\mathrm{u}^2).
\end{aligned}$$

We remark that at this point we have determined the structure of the bimodule $\Omega_D^2\mathcal{A}$. Indeed, if we express R_1 symbolically as

$$\begin{aligned}
&(\boldsymbol{d}A, \boldsymbol{d}W + W \wedge W; \boldsymbol{d}\phi_{12} + A\phi_{12} - \phi_{12}W, \boldsymbol{d}\phi_{21} - \phi_{21}A + W\phi_{21}; \\
&\quad \phi_{12}\phi_{21}, \phi_{21}\phi_{12}),
\end{aligned} \tag{8.31}$$

then $\Omega_D^2\mathcal{A}$ may be identified with the vector space generated by such quantities, i.e.,

$$\Omega_D^2\mathcal{A} \simeq \mathcal{E}^2(M,\mathbb{C}) \oplus \mathcal{E}^2(M,\mathbb{H}) \oplus (\mathcal{E}^1(M,\mathbb{H}))^2 \oplus (C^\infty(M,\mathbb{H}))^2, \tag{8.32}$$

where a typical element may be denoted by $(F, G; \omega_{12}, \omega_{21}; r_{121}, r_{212})$. The algebraic rules for $\Omega_D^2\mathcal{A}$ are then

$$\begin{aligned}
(f,q)\,(F,G;\omega_{12},\omega_{21};r_{121},r_{212}) &= (fF, qG; f\omega_{12}, q\omega_{21}; fr_{121}, qr_{212}), \\
(F,G;\omega_{12},\omega_{21};r_{121},r_{212})\,(f,q) &= (Ff, Gq; \omega_{12}q, \omega_{21}f; r_{121}f, r_{212}q), \\
(F,G;\omega_{12},\omega_{21};r_{121},r_{212})^* &= (F^*, G^*; \omega_{21}^*, \omega_{12}^*; r_{121}^*, r_{212}^*).
\end{aligned} \tag{8.33}$$

The differential $d: \Omega_D^1\mathcal{A} \to \Omega_D^2\mathcal{A}$ is given, in view of (8.26) and (8.27), by

$$\begin{aligned}
d(A, q_{12}, q_{21}, W) = {}&(\boldsymbol{d}A, \boldsymbol{d}W; \boldsymbol{d}q_{12} + A - W, \boldsymbol{d}q_{21} - A + W; \\
&\quad q_{12} + q_{21}, q_{21} + q_{12}),
\end{aligned} \tag{8.34}$$

and the product $\Omega_D^1\mathcal{A} \times \Omega_D^1\mathcal{A} \to \Omega_D^2\mathcal{A}$ by

$$\begin{aligned}
&(A, q_{12}, q_{21}, W)\,(A', q_{12}', q_{21}', W') \\
&\quad = (A \wedge A', W \wedge W'; Aq_{12}' - q_{12}W', Wq_{21}' - q_{21}A'; q_{12}q_{21}', q_{21}q_{12}').
\end{aligned} \tag{8.35}$$

Similarly, $\pi_0(\theta) = R_0 + S_0$, where $S_0 \in \pi_0(dJ_0^1)$ and, on making the replacement $\boldsymbol{m} \mapsto (m_\mathrm{e} \oplus 0)$, we obtain:

$$R_0 = \begin{pmatrix} -c(\boldsymbol{d}A) \otimes I + \lambda \otimes m_{e\perp}^2 & \gamma_5 c(B_{12}) \otimes m_\mathrm{e} & \gamma_5 c(C_{12}) \otimes m_\mathrm{e} \\ \gamma_5 c(B_{12}^*) \otimes m_\mathrm{e} & c(D_1) \otimes I + \tfrac{1}{2}\lambda \otimes m_{e\perp}^2 & c(D_2) \otimes I \\ \gamma_5 c(C_{12}^*) \otimes m_\mathrm{e} & c(D_3) \otimes I & c(D_4) \otimes I + \tfrac{1}{2}\lambda \otimes m_{e\perp}^2 \end{pmatrix}, \tag{8.36}$$

where $1-|\phi_{12}|^2 = \lambda, \boldsymbol{d}\phi_{12}+A\phi_{12}-\phi_{12}W = B_{12}+C_{12}\mathrm{j}, m_{\mathrm{e}\perp}^2 = m_{\mathrm{e}}^2-N_{\mathrm{G}}^{-1}\,\mathrm{tr}(m_{\mathrm{e}}^2)$, and

$$\begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} = \tag{8.37}$$
$$\begin{pmatrix} \boldsymbol{d}W_1 - W_2 \wedge W_2^* & \boldsymbol{d}W_2 + W_1 \wedge W_2 + W_2 \wedge W_1^* \\ -\boldsymbol{d}W_2^* - W_2^* \wedge W_1 - W_1^* \wedge W_2^* & \boldsymbol{d}W_1^* - W_2^* \wedge W_2 \end{pmatrix}.$$

Actually, we have computed R_1 in more generality than necessary, as we have to take into account the relations $A^* = -A$, $W^* = -W$ (i.e., W has no real part), $\phi_{12} = \phi_{21}^*$, as indicated after (8.22). Note that $\phi_{21} =: \Phi_1 + \Phi_2\mathrm{j}$ is identified as the Higgs doublet $\Phi = \binom{\Phi_1}{\Phi_2}$.

If $R := R_0 \oplus (R_1 \otimes I_3)$, the flavourdynamics part of the action for this model would be given by $\mathrm{YM}(\nabla) = \mathrm{YM}(\nabla)_0 + \mathrm{YM}(\nabla)_1$, where

$$\mathrm{YM}(\nabla)_0 = \frac{1}{8\pi^2}\sum_{i,j}\int_M \|(R_0)_{ij}\|^2 = \frac{1}{8\pi^2}\int_M (\mathcal{L}_{20} + \mathcal{L}_{10} + \mathcal{L}_{00})\,\mathrm{d}x,$$

where $\mathcal{L}_{20}, \mathcal{L}_{10}, \mathcal{L}_{00}$ denote the part of the Lagrangian density coming from (ordinary) two-forms, one-forms and zero-forms, respectively; similarly for $\mathrm{YM}(\nabla)_1$. However, here we must pause for a moment to weigh in other considerations. First we wish to consider the color algebra.

8.7. As the action of $\mathcal{B}$ commutes with the off-diagonal terms of the operator (8.16), we have $\Omega_D^1\mathcal{B} \simeq \mathcal{E}^1(M,\mathbb{C}) \oplus \mathcal{E}^1(M,\mathbb{C}^{3\times 3})$ and similarly for $\Omega_D^2\mathcal{A}$. Thus the chromodynamics part $\mathrm{YM}(\nabla)_{\mathrm{c}}$ of the Yang–Mills action is "purely commutative"; we can write in an abbreviated way, for the corresponding curvature R_{c}:

$$R_{\mathrm{c}} = (R_{\mathrm{c}0}, R_{\mathrm{c}1}) = (\boldsymbol{d}A', \boldsymbol{d}K + K \wedge K), \tag{8.38}$$

where A' is a $\mathrm{U}(1)$ gauge field and K is a $\mathrm{U}(3)$ gauge field.

8.8. It is time that we turn to the matter of the "correct" gauge group. The unitary group $\mathcal{U}(\mathcal{A} \otimes \mathcal{B})$ of the algebra $\mathcal{A} \otimes \mathcal{B}$ would obviously be too large. However, the Hilbert space $\mathcal{H}$ is not regarded as an $\mathcal{A} \otimes \mathcal{B}$-module, but rather as an $\mathcal{A}$–$\mathcal{B}$-*bimodule*; this insight comes from the general scheme of matching the algebras with the Poincaré duality mapping. (To get a bimodule, $\mathcal{B}$ must act on $\mathcal{H}$ on the right, rather than on the left; this is easily achieved by replacing the original $\mathcal{B}$ by $\mathcal{B}^{\mathrm{opp}}$, i.e., reversing the order of products in $\mathcal{B}$, which amounts to representing $\mathbb{C} \oplus \mathbb{C}^{3\times 3}$ on $\mathcal{H}$ by right matrix multiplication.)

The relevant symmetries now form the product unitary group $\mathcal{U}(\mathcal{A}) \times \mathcal{U}(\mathcal{B})$, which is still too large. Indeed, the gauge group that we seek is $\mathrm{Map}(M, \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3))$; some sort of unimodularity condition must be imposed in order to extract this as a subgroup of $\mathcal{U}(\mathcal{A}) \times \mathcal{U}(\mathcal{B})$.

We begin by considering the finite-space example again, i.e., $(\mathfrak{h}_F, D_F)$ as an $\mathcal{A}_F$–$\mathcal{B}_F$-bimodule, with $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H}$, $\mathcal{B}_F = \mathbb{C} \oplus \mathbb{C}^{3\times 3}$. Now if

$$u = (\lambda, q; \mu^{-1}, v) \in \mathcal{U}(\mathcal{A}_F) \times \mathcal{U}(\mathcal{B}_F) = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(3), \quad (8.39)$$

and if $\frac{1}{2}(1-\gamma_F) =: e = (0,1)$, $\frac{1}{2}(1+\gamma_F) = 1-e = (1,0)$ are complementary projectors in $\mathcal{A}_F$, we impose the following algebraic chirality condition:

$$u \in \mathrm{SU}(e\mathfrak{h}_F) \times \mathrm{SU}((1-e)\mathfrak{h}_F). \quad (8.40)$$

Since e lies in the centre of $\mathcal{A}_F$, such a u is a direct sum of blocks: $u = eue \oplus (1-e)u(1-e)$. To show that this effects the necessary reduction of the gauge group, we simply compute the determinants of both blocks of u:

$$\begin{aligned} \det(eue) &= \det(q \otimes I_{N_G} \otimes (\mu^{-1} \oplus v)) = (\mu^{-1} \det v)^{N_G}, \\ \det((1-e)u(1-e)) &= \det((\lambda\mu^{-1} I_{N_G}) \oplus (\lambda I_{N_G} \oplus \lambda^* I_{N_G}) \otimes v) \\ &= (\lambda\mu^{-1}(\det v)^2)^{N_G}. \end{aligned} \quad (8.41)$$

Thus (8.40) holds if and only if

$$\lambda^{-1} = \mu = (\det v), \quad (8.42)$$

in which case $u \mapsto (\mu, q, \mu^{-1/3} v)$ gives an isomorphism:

$$(\mathcal{U}(\mathcal{A}_F) \times \mathcal{U}(\mathcal{B}_F)) \cap (\mathrm{SU}(e\mathfrak{h}_F) \times \mathrm{SU}((1-e)\mathfrak{h}_F)) \simeq \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3). \quad (8.43)$$

It is clear that this mechanism cannot be incorporated into the GWS model of section 7, since the relations (8.42) require the presence of the bimodule structure provided by the second algebra $\mathcal{B}$. Notice also that condition (8.40) imposes no restriction on the number of generations N_G.

The restriction of the representation π to this subgroup gives linear relations between generators of the Lie algebra, whose coefficients may be identified with the hypercharges. In fact, one may regard μ as the parameter for the symmetry group $\mathrm{U}(1)_Y$. Writing $v = \mu^{1/3} v_0$, where $v_0 \in \mathrm{SU}(3)$ and $\mu^{1/3}$ is a cube root of μ, we get

$$\begin{aligned} \pi(u) &= \begin{pmatrix} \lambda & 0 \\ 0 & q \end{pmatrix} \mu^{-1} \oplus \begin{pmatrix} \lambda \oplus \lambda^* & 0 \\ 0 & q \end{pmatrix} \otimes \mu^{1/3} v_0 \\ &= \begin{pmatrix} \mu^{-2} & 0 \\ 0 & q\mu^{-1} \end{pmatrix} \oplus \begin{pmatrix} \mu^{-2/3} \oplus \mu^{4/3} & 0 \\ 0 & q\mu^{1/3} \end{pmatrix} \otimes v_0, \end{aligned} \quad (8.44)$$

which is interpreted as assigning hypercharges of -1 to e_L and ν_L, -2 to e_R, $-\frac{2}{3}$ to d_R, $\frac{4}{3}$ to u_R, and $\frac{1}{3}$ to d_L and u_L. We see that these are indeed the correct hypercharges for the component fermions of the Standard Model [23]. The total hypercharge will be zero, as is needed for full anomaly cancellation in the quantized theory [18].

To complete our task, we further digress in order to justify (8.40) in the general case. We remark that e and $(1-e)$ form a basis for the centre of the algebra

$\mathcal{A}_F$, so that (8.40) says that u is unimodular with respect to the determinant function associated to any of the following traces:

$$\tau^{(c)}(x) := \operatorname{Tr}(cx), \quad \text{for } x \in \mathcal{A}_F \otimes \mathcal{B}_F, \tag{8.45}$$

where c runs over self-adjoint elements of the centre of $\mathcal{A}_F$. The associated determinant function is given by

$$\det{}^{(c)}(\exp x) := \exp(\tau^{(c)}(x)). \tag{8.46}$$

The phase of this determinant function can be computed by

$$\operatorname{Phase}^{(c)}(u) := \frac{1}{2\pi \mathrm{i}} \int_0^1 \tau^{(c)}(u'(t)u(t)^{-1})\,\mathrm{d}t, \tag{8.47}$$

where $t \mapsto u(t)$ is a smooth path from 1 to u in $\mathcal{U}(\mathcal{A}_F) \times \mathcal{U}(\mathcal{B}_F)$. Thus (8.40) is equivalent to the following condition:

$$\operatorname{Phase}^{(c)}(u) = 0 \quad \text{for all self-adjoint } c \in Z(\mathcal{A}_F). \tag{8.48}$$

We can now formulate the sought-after unimodularity condition in general. If $(\mathcal{H}, D)$ is a d^+-summable K-cycle over $\mathcal{A} \otimes \mathcal{B}$, which is an $\mathcal{A}$–$\mathcal{B}$-bimodule, then (8.48) makes perfect sense provided (a) u lies in the identity component of $\mathcal{U}(\mathcal{A}) \times \mathcal{U}(\mathcal{B})$; and (b) the traces on the right of (8.47) are obtained from the Dixmier trace on $\mathcal{L}^{1+}(\mathcal{H})$ by

$$\tau^{(c)}(x) := \operatorname{Tr}^+(\pi(cx)|D|^{-n}), \tag{8.49}$$

where c runs over self-adjoint elements of the centre of $\mathcal{A}$. Let $\mathcal{SU}(\mathcal{A}, \mathcal{B})$ denote the subgroup of $\mathcal{U}(\mathcal{A}) \times \mathcal{U}(\mathcal{B})$ whose elements satisfy (8.48). We adopt this as our definition of the *gauge group* (of the second kind) for our model. Using the trace theorem once more, it can then be seen that $\mathcal{SU}(\mathcal{A}, \mathcal{B}) = \operatorname{Map}(M, \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3))$. At the infinitesimal level, the unimodularity condition (8.42) gives the following reduction of the gauge fields:

$$A = A' = -(K_{11} + K_{22} + K_{33}). \tag{8.50}$$

8.9. As persuasively argued by Connes and Lott [12], to compute $\mathrm{YM}(\nabla)_0 + \mathrm{YM}(\nabla)_1 + \mathrm{YM}(\nabla)_\mathrm{c}$ would be surely irrelevant, as the Hilbert space of the fermions is not irreducible under the action of the gauge group; in effect, we would be artificially imposing relations between the coupling constants. A more general gauge invariant bosonic action can be obtained by multiplying R_0, R_1, $R_{\mathrm{c}0}$, $R_{\mathrm{c}1}$ by (arbitrary) positive operators z_0, z_1, $z_{\mathrm{c}0}$, $z_{\mathrm{c}1}$, commuting with the actions of $\mathcal{A}$ and $\mathcal{B}$, before taking the Dixmier trace. On the other hand, Kastler [29] will only admit $z_0 = z_{\mathrm{c}0} = \alpha_l I$ and $z_1 = z_{\mathrm{c}1} = \alpha_q I$ with $\alpha_l + \alpha_q = 1$. We leave it to the reader to sort out the different possibilities, according to his judgement and taste; for the "maximalist" view on the relations among the constants appearing in the Lagrangian, whereupon one obtains all parameters of the Standard Model from the fermion masses and a single universal coupling constant, we refer to ref. [30]. At any rate, if we follow Connes and Lott, it is clear that we have reproduced the terms in the bosonic part of the standard model Lagrangian, with arbitrary constants—if we were allowed to rescale the Higgs field *ad libitum*. More precisely, we get:

1. The terms corresponding to the gauge fields: $\|\boldsymbol{d}A\|^2$, $\|\boldsymbol{d}W + W \wedge W\|^2$, $\|\boldsymbol{d}K + K \wedge K\|^2$ (the electroweak and the gluon part, respectively).

2. The kinetic term for the Higgs field. For later use we give a "precise" coefficient here. We had

$$\mathrm{Tr}^+(R_1^2 \otimes I_3) = 3\,\mathrm{Tr}^+(R_1^2) = \frac{1}{8\pi^2}\int_M \mathcal{L}_{21} + \mathcal{L}_{11} + \mathcal{L}_{01}\,\mathrm{d}x\,, \qquad (8.51)$$

and from (8.30) we compute

$$\mathcal{L}_{11} = 6\,\mathrm{tr}(m_\mathrm{d} m_\mathrm{d}^\dagger + m_\mathrm{u}^2)\|\boldsymbol{d}\phi_{12} + A\phi_{12} - \phi_{12}W\|^2 =: C_q\|\boldsymbol{d}\phi_{12} + A\phi_{12} - \phi_{12}W\|^2 \qquad (8.52)$$

and therefore

$$\begin{aligned}\mathcal{L}_{10} &= 2\,\mathrm{tr}(m_\mathrm{e}^2)\|\boldsymbol{d}\phi_{12} + A\phi_{12} - \phi_{12}W\|^2 \\ &=: C_l\|\boldsymbol{d}\phi_{12} + A\phi_{12} - \phi_{12}W\|^2. \end{aligned} \qquad (8.53)$$

3. The Higgs self-interaction term. Proceeding as above, we get

$$\mathcal{L}_{01} = D_q(1 - |\Phi|^2)^2\,, \qquad (8.54)$$

where

$$D_q = \tfrac{9}{2}\,\mathrm{tr}(m_\mathrm{d} m_\mathrm{d}^\dagger)^2 + \tfrac{9}{2}\,\mathrm{tr}(m_\mathrm{u}^4) + 3\,\mathrm{tr}(m_\mathrm{d} m_\mathrm{d}^\dagger m_\mathrm{u}^2) - 3(\mathrm{tr}(m_\mathrm{d} m_\mathrm{d}^\dagger + m_\mathrm{u}^2))^2/N_\mathrm{G}. \qquad (8.55)$$

Also

$$\mathcal{L}_{00} = D_l(1 - |\Phi|^2)^2, \qquad (8.56)$$

where

$$D_l = \tfrac{3}{2}(\mathrm{tr}(m_\mathrm{e}^4) - (\mathrm{tr}\,m_\mathrm{e}^2)^2)/N_\mathrm{G}. \qquad (8.57)$$

One must now make the Wick rotation and can write, for instance,

$$A = \mathrm{i}(g_1/2)B_\mu\,\mathrm{d}x^\mu\,, \qquad W = -\mathrm{i}g_2 W_\mu^a(\tau_a/2)\,\mathrm{d}x^\mu\,,$$

where the τ_a are the Pauli matrices..., to translate into the physicist's language.

We close this subsection by mentioning that Chamseddine et al. [6] have recently developed a formalism apparently akin to noncommutative geometry and used it to obtain Lagrangians for grand unification models.

8.10. The fermionic action is $\langle\psi \mid D_\nabla\psi\rangle$, where

$$D_\nabla = D - \mathrm{i}\pi(\nabla) = \begin{pmatrix} (\not\partial + \not A)\otimes I & \gamma_5 \boldsymbol{m}^\dagger\phi_{12} \\ \gamma_5\phi_{21}\boldsymbol{m} & (\not\partial + \not W)\otimes I \end{pmatrix}. \qquad (8.58)$$

Clearly, as in section 7, the Yukawa part is the contribution of the off-diagonal terms in D_∇ and (again imposing chirality after the Wick rotation) we obtain an expression altogether analogous to the fermionic part of the Lagrangian of the Standard Model. We shall not bother to write it except for pointing out that the terms with masses m_d correspond to those of the GWS Lagrangian, while the new terms with masses m_u like $\bar{u}_\mathrm{L}\phi_1^* m_\mathrm{u} u_\mathrm{R}, \bar{u}_\mathrm{R}\phi_1 m_\mathrm{u} u_\mathrm{L}$, etc. are of the same form but with the Higgs doublet $\phi_{21} = \Phi$ replaced by $\tilde{\Phi}$.

Now, however, a new consideration enters the picture. If Yukawa terms of the above type are to represent (in the broken symmetry phase) the mass terms of fermions in the Standard Model, then we are not allowed to scale the Higgs field arbitrarily. The net result is a relationship between the two parameters of the Higgs field, leading immediately to the formula:

$$m_\mathrm{H} = 2\sqrt{\frac{D_l + D_q}{C_l + C_q}} \tag{8.59}$$

for the mass of the Higgs particle. We recall that the values of D_l, D_q, C_l, C_q given in formulae (8.52) to (8.57) are only indicative, as the Yang–Mills functional could have been chosen somewhat differently. As they stand, however, they illustrate the conclusion that the relationship implies that the Higgs mass is of the same order of magnitude as the top quark mass.

In fine, all the properties of the standard model Lagrangian may be obtained from a single K-cycle in the framework of Connes' noncommutative geometry; moreover, it is "predicted" that the Higgs mass will fall well within the perturbative regime. The particular relation for the Higgs mass, however, as every other relation purported to follow from noncommutative geometry, is washed out by the standard renormalization process [1].

Appendix A. Clifford algebras, spinor bundles, Dirac operators and all that

A.1. In this appendix we collect the facts about Clifford algebras and spinors that we need. General references are the fundamental paper [2], the books [3, 5, 31] and the useful surveys [15, 20].

The *Clifford algebra* $\mathrm{Cl}(E) = \mathrm{Cl}(E,q)$ determined by a real vector space E equipped with a quadratic form q is an associative algebra generated by the elements of E subject to the relation $x \cdot x = -q(x,x)$. It may be defined as $\mathrm{Cl}(E,q) := \mathcal{T}(E)/I(q)$, where $\mathcal{T}(E)$ is the tensor algebra over E and $I(q)$ is the ideal generated by $\{ x \otimes x + q(x,x) : x \in E \}$. Also denote by q the bilinear form $q(x,y) := \frac{1}{2}(q(x+y,x+y) - q(x,x) - q(y,y))$. The canonical mapping of E into $\mathrm{Cl}(E)$ is injective, so E may be regarded as a subspace of $\mathrm{Cl}(E)$; we then have the relation

$$xy + yx = -2q(x,y) \quad \text{for } x,y \in E. \tag{A.1}$$

It is clear that $\mathrm{Cl}(E)$ satisfies a universal property:

Proposition A.1. *If B is a real algebra with identity and $f: E \to B$ is a linear mapping such that $f(x)^2 = q(x,x)$, then it factors uniquely through $\mathrm{Cl}(E)$: $f = \tilde{f}|_E$, where $\tilde{f}: \mathrm{Cl}(E) \to B$ is an algebra homomorphism.* □

From this one sees that the orthogonal group of q consists of automorphisms of the Clifford algebra: $O(E,q) \subset \mathrm{Aut}(\mathrm{Cl}(E))$. In particular, the orthogonal transformation $x \mapsto -x$ yields an involutive automorphism of $\mathrm{Cl}(E)$, which we denote by α; explicitly, $\alpha(x_1 \cdots x_k) = (-)^k x_1 \cdots x_k$. Its ± 1 eigenspaces give a $\mathbb{Z}_2$-grading $\mathrm{Cl}^0(E) \oplus \mathrm{Cl}^1(E)$ of $\mathrm{Cl}(E)$.

Taking now for B the opposite algebra of $\mathrm{Cl}(E)$ (the same vector space with the product reversed), we also find an involutive anti-automorphism called β, i.e., $\beta(x_1 \cdots x_k) = x_k \cdots x_1$. Note that α and β commute. We define another anti-automorphism $x \mapsto \bar{x}$, called conjugation, by $\bar{x} := \beta(\alpha(x))$.

Given two $\mathbb{Z}_2$-graded algebras A, B, let $A \hat{\otimes} B$ denote their $\mathbb{Z}_2$-*graded* tensor product:

$$
\begin{aligned}
(A \hat{\otimes} B)^0 &:= (A^0 \otimes B^0) \oplus (A^1 \otimes B^1), \\
(A \hat{\otimes} B)^1 &:= (A^1 \otimes B^0) \oplus (A^0 \otimes B^1),
\end{aligned} \tag{A.2}
$$

with multiplication

$$
(x \otimes y) \cdot (z \otimes w) = (-)^{\deg y \deg z}(xz \otimes yw). \tag{A.3}
$$

Many properties of the Clifford algebra come from the following simple proposition.

Proposition A.2 (Chevalley). *Let q_1, q_2 be quadratic forms on real vector spaces E_1, E_2, respectively; define $f: E_1 \oplus E_2 \to \mathrm{Cl}(E_1, q_1) \hat{\otimes} \mathrm{Cl}(E_2, q_2)$ by $(x,y) \mapsto x \otimes 1 + 1 \otimes y$. Then f extends uniquely to a $\mathbb{Z}_2$-graded isomorphism of $\mathrm{Cl}(E_1 \oplus E_2, q_1 \oplus q_2)$ onto $\mathrm{Cl}(E_1, q_1) \hat{\otimes} \mathrm{Cl}(E_2, q_2)$.*

Proof. Check that $f(x,y)^2 = -(q_1 \oplus q_2)(x \oplus y, x \oplus y)\, 1 \otimes 1$, using (A.3). It follows that there is an algebra homomorphism $\tilde{f}: \mathrm{Cl}(E_1 \oplus E_2, q_1 \oplus q_2) \to \mathrm{Cl}(E_1, q_1) \hat{\otimes} \mathrm{Cl}(E_2, q_2)$. Since the $x \otimes 1 + 1 \otimes y$ generate $\mathrm{Cl}(E_1, q_1) \hat{\otimes} \mathrm{Cl}(E_2, q_2)$ as an algebra, $\tilde{f}$ is onto; and one easily sees that it is injective by examining its effect on a basis for $\mathrm{Cl}(E_1 \oplus E_2, q_1 \oplus q_2)$ generated by bases for E_1 and E_2. □

Corollary A.3. *If $e_1, e_2, \dots, e_n$ is a basis of E, then 1 and $\{ e_{i_1} e_{i_2} \cdots e_{i_k} : 1 \leq k \leq n,\ 1 \leq i_1 < \cdots < i_k \leq n \}$ form a basis of $\mathrm{Cl}(E)$. Thus* $\dim \mathrm{Cl}^0(E) = \dim \mathrm{Cl}^1(E) = 2^{n-1}$.

Proof. If $\dim E = 1$, we have $\mathcal{T}(E) \simeq \mathbb{R}[X]$, a polynomial algebra. Then $\mathrm{Cl}(E) \simeq \mathbb{R}[X]/(X^2 + q(X,X))$, with basis $\{1, X\}$. Since there is a q-orthogonal decomposition $E = \bigoplus_{i=1}^{n} E_i$ into one-dimensional subspaces, the result follows by induction on n. □

The natural filtration of the tensor algebra induces a filtration on the underlying vector space of $\mathrm{Cl}(E)$; in the same way one gets the usual filtration of the exterior algebra $\Lambda^\bullet(E)$. There is a canonical vector space isomorphism $\Xi: \Lambda^\bullet(E) \to \mathrm{Cl}(E)$, compatible with the filtrations, given by

$$\Xi_k(x_1 \wedge \cdots \wedge x_k) := \frac{1}{k!} \sum_{\tau \in S_k} \mathrm{sign}(\tau)\, x_{\tau(1)} \cdots x_{\tau(k)}. \tag{A.4}$$

Let $C^0 \subset C^1 \subset \cdots \subset C^n = \mathrm{Cl}(E)$ be the first filtration. We can define an *associated graded algebra* G by $G := \bigoplus_k C^k/C^{k-1}$. Composing Ξ_k with the canonical projection $C^k \to C^k/C^{k-1}$, we obtain a graded algebra isomorphism between $\Lambda^\bullet(E)$ and G.

A.2. The multiplicative group of units $\mathrm{Cl}(E)^u$ is an open subset of $\mathrm{Cl}(E)$, since x is a unit iff $y \mapsto xy$ is a nonsingular linear transformation; hence it is a Lie group. The *twisted adjoint representation* of $\mathrm{Cl}(E)^u$ on $\mathrm{Cl}(E)$ is defined by $y \mapsto \varphi(x)y := \alpha(x)yx^{-1}$. Let Γ be the subgroup $\{x \in \mathrm{Cl}(E)^u : \varphi(x)y \in E \text{ for all } y \in E\}$; Γ is invariant under α and β. The map $N: \mathrm{Cl}(E) \to \mathrm{Cl}(E) : x \mapsto x\bar{x}$ is called the *spinorial norm* of x; for $x \in E$, we have $N(x) = q(x,x)$.

Henceforth, we concentrate on the case $E = \mathbb{R}^n$, $q_n(x,x) := (x^1)^2 + \cdots + (x^n)^2$. Write Cl_n for $\mathrm{Cl}(\mathbb{R}^n, q_n)$. It is clear that $\mathrm{Cl}_1 \simeq \mathbb{C}$, $\mathrm{Cl}_2 \simeq \mathbb{H}$ (the quaternions). We write Γ_n for the subgroup Γ in this case.

Proposition A.4.

(a) *The kernel of* $\varphi: \Gamma_n \to GL(n,\mathbb{R})$ *is* $\mathbb{R}^*$, *the nonzero multiples of 1.*

(b) *The restriction of* N *to* Γ_n *is a group homomorphism into* $\mathbb{R}^*$ *and* $N \circ \alpha = N$.

(c) $\varphi(\Gamma_n) \subset \mathrm{O}(n)$, *the orthogonal group of* $\mathbb{R}^n$.

(d) *For* $x \in \mathbb{R}^n \setminus \{0\}$, $x \in \Gamma_n$ *and* $\varphi(x)$ *is the reflection across the hyperplane orthogonal to* x.

(e) *Let* $\mathrm{Pin}(n)$ *denote the kernel of* $N : \Gamma_n \to \mathbb{R}^*$. *The restriction of* φ *to* $\mathrm{Pin}(n)$ *is a surjection onto* $\mathrm{O}(n)$ *with kernel* $\{1,-1\}$. *In other words, the following sequence is exact:*

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Pin}(n) \longrightarrow \mathrm{O}(n) \longrightarrow 0. \tag{A.5}$$

Proof. If $u \in \ker\varphi$, let $u = u_0 + u_1$ with $u_i \in \mathrm{Cl}_n^i$ $(i = 0,1)$; then $u_i x = (-)^i x u_i$ for $x \in \mathbb{R}^n$. If $\{e_1,\dots,e_n\}$ is an orthogonal basis for $\mathbb{R}^n$, we can write $u_0 = z_0 + e_1 z_1$ with $z_i \in \mathrm{Cl}_{n-1}^i$. Now $e_1 z_0 - z_1 = e_1 u_0 = u_0 e_1 = e_1 z_0 + z_1$, so $z_1 = 0$. By permuting the basis of $\mathbb{R}^n$, we see that u_0 does not involve any e_j, and so is a scalar. A similar argument shows that u_1 vanishes, so $u \in \mathbb{R}^*$.

Now suppose $x \in \Gamma_n$; then $N(x) \in \Gamma_n$ too. For $y \in \mathbb{R}^n$, one has $\varphi(\bar{x})y = \beta(\varphi(\bar{x})y) = \beta(\alpha(\bar{x})y(\bar{x})^{-1}) = \alpha(x)^{-1}yx$. From this, $y = \alpha(N(x))y \times N(x)^{-1}$, which implies $N(x) \in \mathbb{R}^*$. Moreover, for $x, y \in \Gamma_n$ one has $N(xy) = xy\bar{y}\bar{x} = N(x)N(y)$. That $N \circ \alpha = N$ is obvious. For the orthogonality, it is enough to note that $N(\varphi(x)y) = N(\alpha(x))N(y)N(x^{-1}) = N(y)$.

If $x, y \in \mathbb{R}^n$, $x \neq 0$, from (A.1) we get

$$\varphi(x)y = -xyx^{-1} = y - 2\frac{q(x,y)}{q(x,x)}x.$$

Since the reflections generate the whole group $\mathrm{O}(n)$, $\varphi\colon \mathrm{Pin}(n) \to \mathrm{O}(n)$ is surjective. Also, $\ker\varphi \cap \ker N = \{1, -1\}$. □

Note that $\mathrm{Pin}(n)$ is a closed subgroup of the group of units in Cl_n, thus carries a natural Lie group structure. This makes $\varphi|_{\mathrm{Pin}(n)}$ a Lie group homomorphism. The group $\mathrm{Spin}(n)$ is by definition the pre-image of $\mathrm{SO}(n)$ under φ. We have the exact sequence:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(n) \xrightarrow{\varphi} \mathrm{SO}(n) \longrightarrow 0. \tag{A.6}$$

If $x \in \mathrm{Pin}(n)$, then $\varphi(x)$ can be written as a composition of reflections $\sigma_1 \cdots \sigma_m$. Pick elements $x_j \in \mathbb{R}^n \cap \mathrm{Pin}(n)$ such that $\varphi(x_j) = \sigma_j$ for each j. By proposition A.4(e), $x = \pm x_1 \cdots x_m$. Thus $\mathrm{Pin}(n)$ is the disjoint union of $\mathrm{Pin}(n) \cap \mathrm{Cl}_n^0 = \mathrm{Spin}(n)$ and $\mathrm{Pin}(n) \cap \mathrm{Cl}_n^1$. For example, $\mathrm{Cl}_1 = \mathbb{C}$, $\mathrm{Cl}_1^0 = \mathbb{R}$ and $\mathrm{Cl}_1^1 = \mathrm{i}\mathbb{R}$. Here α is complex conjugation and β is the identity; N is the square of the usual norm on $\mathbb{C}$, $\Gamma_1 = \{ z \in \mathbb{C}^* : \bar{z}iz^{-1} \subset \mathrm{i}\mathbb{R} \} = \mathbb{R} \cup \mathrm{i}\mathbb{R} \setminus \{0\}$ and $\mathrm{Pin}(1) = \{1, \mathrm{i}, -1, -\mathrm{i}\}$, $\mathrm{Spin}(1) = \{1, -1\}$.

Proposition A.5. *For $n \geq 2$, $\varphi\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a connected twofold covering. For $n \geq 3$, this is the universal covering group of $\mathrm{SO}(n)$.*

Proof. Since $\mathrm{SO}(n)$ is connected, the connectedness of $\mathrm{Spin}(n)$ follows from finding a path in $\mathrm{Spin}(n)$ joining the two elements $\{\pm 1\}$ of $\ker\varphi$. Such a path is given by

$$t \mapsto \cos\pi t + \sin\pi t\, e_1 e_2 = (\cos\tfrac{1}{2}\pi t\, e_1 + \sin\tfrac{1}{2}\pi t\, e_2)(-\cos\tfrac{1}{2}\pi t e_1 + \sin\tfrac{1}{2}\pi t\, e_2),$$

where $\{e_1, e_2\}$ are the first two elements of an orthogonal basis of $\mathbb{R}^n$. This twofold covering is the universal covering since the fundamental group of $\mathrm{SO}(n)$ is $\mathbb{Z}_2$ for $n \geq 3$. □

A.3. Write $\mathrm{Cl}'(n) := \mathrm{Cl}(\mathbb{R}^n, -q_n)$. Note that $\mathrm{Cl}'_2 \simeq \mathbb{R}^{2\times 2}$. There are isomorphisms of real algebras $\mathrm{Cl}_n \otimes \mathrm{Cl}'_2 \simeq \mathrm{Cl}'_{n+2}$ and $\mathrm{Cl}'_n \otimes \mathrm{Cl}_2 \simeq \mathrm{Cl}_{n+2}$. Indeed, if

$e_1, \dots, e_n$ form the canonical basis of $\mathbb{R}^n$, denote by $e'_1, \dots, e'_n$ their images in Cl'_n. We can define a linear mapping $\psi: \mathbb{R}^{n+2} \to \mathrm{Cl}_n \otimes \mathrm{Cl}'_2$ by

$$\psi(e_1) := 1 \otimes e'_1, \qquad \psi(e_2) := 1 \otimes e'_2,$$
$$\psi(e_i) := e_{i-2} \otimes e'_1 e'_2 \quad \text{for } 3 \le i \le n+2. \tag{A.7}$$

One checks that $\psi(e_i)\psi(e_j) + \psi(e_i)\psi(e_j) = 2\delta_{ij}(1 \otimes 1)$ in all cases. By the universal property, ψ extends to a homomorphism of Cl'_{n+2} into $\mathrm{Cl}_n \otimes \mathrm{Cl}'_2$, which is clearly onto and must be injective by dimension count. The argument establishing $\mathrm{Cl}'_n \otimes \mathrm{Cl}_2 \simeq \mathrm{Cl}_{n+2}$ is similar.

The *complexification* $\mathbb{C}\mathrm{l}(E,q) := \mathrm{Cl}(E,q) \otimes_{\mathbb{R}} \mathbb{C}$ can be regarded as the Clifford algebra over $\mathbb{C}$ of the complexified vector space $E_{\mathbb{C}} := E \otimes_{\mathbb{R}} \mathbb{C}$ with the complexified quadratic form $q_{\mathbb{C}}$. On $\mathbb{C}^n$ all nondegenerate bilinear forms are equivalent; in particular $\mathbb{C}\mathrm{l}(E,q_n) \simeq \mathbb{C}\mathrm{l}'(E,q_n) =: \mathbb{C}\mathrm{l}_n$. Thus we obtain the following classification of complex Clifford algebras.

Theorem A.6. *The complex algebras $\mathbb{C}\mathrm{l}_n$ are isomorphic to $\mathbb{C}^{2^k \times 2^k}$ for $n = 2k$ and to $\mathbb{C}^{2^k \times 2^k} \oplus \mathbb{C}^{2^k \times 2^k}$ for $n = 2k+1$.* □

We see that $\mathbb{C}\mathrm{l}_{2k}$ is simple and its unique simple module has dimension 2^k, whereas $\mathbb{C}\mathrm{l}_{2k+1}$ has two simple modules, each of dimension 2^k. Note that there is an isomorphism of algebras: $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}^{2\times 2}$.

For the classification of real Clifford algebras, which is more involved, see ref. [31, §1.4].

We will also find useful the group $\mathrm{Spin}^c(n)$, defined as $(\mathrm{Spin}(n) \times \mathrm{U}(1))/\mathbb{Z}_2$, where we quotient by the relation $(h,z) \sim (-h,-z)$. We have a homomorphism $\varphi^c: \mathrm{Spin}^c(n) \to \mathrm{SO}(n) \times \mathrm{U}(1)$ given by $\varphi^c(x,\lambda) := (\varphi(x), \lambda^2)$, so that the following sequence is exact:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^c(n) \xrightarrow{\varphi^c} \mathrm{SO}(n) \times \mathrm{U}(1) \longrightarrow 0. \tag{A.8}$$

Note that φ^c is a representation of $\mathrm{Spin}^c(n)$ on $\mathbb{C}^n = \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C}$.

A.4. Let E be a real vector space of even dimension $n = 2k$. A (complex) spinor space associated to (E,q) is simply a $\mathbb{C}\mathrm{l}(E,q)$-module. There is no canonically defined spinor space; one can give an explicit representation by choosing a complex structure J on E (i.e., J is an isometry such that $J^2 = -\mathrm{id}$). On (E,J), viewed as a complex space by the usual recipe $(\alpha + i\beta)x := \alpha x + \beta J x$, we can define the nondegenerate hermitian form

$$(u \mid v) := q(u,v) + iq(Ju,v), \tag{A.9}$$

naturally extended to the spinor Fock space $F(E,J) := \bigoplus_{r=0}^{k} \Lambda^r_{\mathbb{C}}(E,J)$. Then there is an isomorphism of algebras $c: \mathbb{C}\mathrm{l}(E,q) \to \mathrm{End}_{\mathbb{C}} F(E,J)$, defined as follows. If $x \in E$, define $e(x): s \mapsto x \wedge s$, and let $i(x)$ be contraction with the bra vector $(x|$, given by

$$i(x)\,(x_1 \wedge \cdots \wedge x_k) := \sum_{j=1}^{k} (-)^{j-1} (x \mid x_j)\, x_1 \wedge \cdots \wedge \widehat{x}_j \wedge \cdots \wedge x_k. \qquad (A.10)$$

One easily checks the anticommutators $\{e(x), e(y)\} = \{i(x), i(y)\} = 0$ and $\{e(x), i(y)\} = (y \mid x)$, so that

$$\{e(x) - i(x), e(y) - i(y)\} = -(y \mid x) - (x \mid y) = -2q(x,y). \qquad (A.11)$$

We write $c(x) := e(x) - i(x)$, and recover $c(x)c(y) + c(y)c(x) = -2q(x,y)$. Thus the linear map $c: E \to \mathrm{End}_{\mathbb{C}} F(E,J)$ extends to an injective algebra homomorphism from $\mathbb{C}\mathrm{l}(E,q)$ to $\mathrm{End}_{\mathbb{C}} F(E,J)$, which is surjective by dimension count. We may therefore identify $\mathbb{C}\mathrm{l}(E,q)$ with its image under c. This is just the theory of the canonical anticommutation relations, where $e(x)$, $i(x)$ are regarded as creation and annihilation operators, respectively.

Remark. An analogous construction works for real Clifford algebras [20, 31]; one need only replace $(x \mid x_j)$ by $q(x, x_j)$ in (A.10). Subject to this understanding, we can also represent the Clifford product by $x \in \mathrm{Cl}_n$ as $e(x) - i(x)$ acting on $\Lambda^\bullet \mathbb{R}^n$.

The Fock space is $\mathbb{Z}_2$-graded by parity of the order of exterior products. Now for each $x \in E$, $c(x)$ is an odd endomorphism, i.e., it exchanges $F_+(E,J)$ and $F_-(E,J)$, and so c is a graded isomorphism from $\mathbb{C}\mathrm{l}(E,q)$ to $\mathrm{End}_{\mathbb{C}} F(E,J)$.

If $x \in E$ and $s, s' \in F(E,J)$, one has $(s \mid e(x)s') = (i(x)s \mid s')$, from which we conclude that every $c(x)$ is skew-adjoint on $F(E,J)$. Hence we have

$$(c(x)s \mid c(x)s') = N(x)(s \mid s'). \qquad (A.12)$$

One has the inclusions $\mathrm{Spin}(n) \subset \mathrm{Pin}(n) \subset \mathrm{Cl}_n \subset \mathbb{C}\mathrm{l}_n$, with $\mathrm{Spin}(n) \subset \mathbb{C}\mathrm{l}_n^0$. The (complex) *spin representation* of $\mathrm{Spin}(n)$ is simply the restriction of c to $\mathrm{Spin}(n)$. The restriction to $\mathrm{Pin}(n)$ is unitary in view of (A.12), and irreducible, because the complex subalgebra generated by this group is all of $\mathbb{C}\mathrm{l}_n$. The restriction to $\mathrm{Spin}(n)$ is the direct sum of two irreducible "half-spin" representations, the nonisomorphic simple modules being $F_+(E,J)$ and $F_-(E,J)$.

We also note that $\mathrm{Spin}^c(n)$ embeds naturally in $\mathbb{C}\mathrm{l}_n$ by $(x, \lambda) \mapsto \lambda x$ for $x \in \mathrm{Spin}(n)$, $\lambda \in \mathrm{U}(1)$. We thus obtain a unitary representation of $\mathrm{Spin}^c(n)$ on $F(E,J)$ by restriction of c.

A.5. Consider the element

$$\gamma := \mathrm{i}^{\lceil n/2 \rceil} e_1 e_2 \cdots e_n \in \mathbb{C}\mathrm{l}_n, \qquad (A.13)$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis for (E, q_n), and $\lceil n/2 \rceil = k$ for $n = 2k$ or $n = 2k - 1$. If $e_i' = \sum_{j=1}^n g_{ij} e_j$ is another orthonormal basis, then

$\mathrm{i}^{\lceil n/2 \rceil} e'_1 e'_2 \cdots e'_n = \det(g)\gamma = \pm\gamma$, so γ is independent of the basis provided the vector space E is given a fixed orientation; this we will assume. It is clear that $\gamma^2 = 1$, and that

$$x\gamma = (-)^{n-1}\gamma x \quad \text{for } x \in E, \tag{A.14}$$

since each e_j anticommutes with every factor of γ but itself.

In particular, if n is odd, then γ lies in the two-dimensional centre of $\mathbb{C}\mathrm{l}(n)$. If n is even, then $x\gamma = -\gamma x$ for $x \in E$. We shall call γ the *chirality element* of $\mathbb{C}\mathrm{l}_n$.

Suppose that $n = 2k$ is even. Then $p^+ := \frac{1}{2}(1+\gamma)$ and $p^- := \frac{1}{2}(1-\gamma)$ are complementary idempotents in $\mathbb{C}\mathrm{l}_n$, i.e., $(p^\pm)^2 = p^\pm$, $p^+p^- = p^-p^+ = 0$, and $p^+ + p^- = 1$. If S is any left module for $\mathbb{C}\mathrm{l}_n$, it is $\mathbb{Z}_2$-graded: $S = S^+ \oplus S^-$, where $S^\pm = p^\pm S$ are the ± 1 eigenspaces for $s \mapsto c(\gamma)s$.

In particular, the spinor Fock space $F(E,J)$ is $\mathbb{Z}_2$-graded. An orthonormal basis for the Hilbert space (E,J) is given by $\{w_1, \dots, w_k\}$, where $w_j := (e_{2j-1} - Je_{2j})/\sqrt{2}$; write also $w_j^* := (e_{2j-1} + Je_{2j})/\sqrt{2}$. Then $w_j^* w_j - w_j w_j^* = -2\mathrm{i} \times e_{2j-1}e_{2j}$ in $\mathbb{C}\mathrm{l}(E)$, so that $\gamma = 2^{-n/2}(w_1^* w_1 - w_1 w_1^*) \cdots (w_k^* w_k - w_k w_k^*)$. Since $(w_i \mid w_j^*) = 0$ for any i, j, one checks that $c(\gamma)$ acts by $(-1)^r$ on $\Lambda^r_{\mathbb{C}}(E,J)$; thus $S^+ = F_+(E,J)$ and $S^- = F_-(E,J)$. The grading operator $c(\gamma)$ is the *chirality operator* on the spinor Fock space.

A.6. The trace of the spin representation induces a canonical inner product on $\mathbb{C}\mathrm{l}_n$ (for $n = 2k$ even). If $e_K := e_{k_1} \cdots e_{k_r}$ for $K = \{k_1, \dots, k_r\} \subset \{1, \dots, n\}$ (with indices increasingly ordered), then $\mathrm{tr}(c(e_K)) = 0$ for odd K since $c(e_K)$ interchanges S^+ and S^-, whereas $2\,\mathrm{tr}(c(e_K)) = \mathrm{tr}([c(e_{k_1}), c(e_{K\setminus k_1})]) = 0$ for K even and nonvoid. Hence $\mathrm{tr}(c(x)) = 2^{n/2} a_\emptyset$ is where $a_\emptyset$ is the scalar term in the expansion $x = \sum_K a_K e_K$ of $x \in \mathbb{C}\mathrm{l}_n$. Now if $y = \sum_K b_K e_K \in \mathbb{C}\mathrm{l}_n$, then since each $c(e_i)$ is skew-adjoint, we get

$$\mathrm{tr}(c(x)^\dagger c(y)) = \sum_{K,L} a_K^* b_L (-)^r \,\mathrm{tr}(c(\beta(e_K)e_L)) = 2^{n/2} \sum_K a_K^* b_K. \tag{A.15}$$

In particular, if $x = e_1 \cdots e_n$, this trace gives the coefficient of $e_1 \cdots e_n$ in the expansion of y.

A.7. Let M be a compact manifold without boundary. We consider metric vector bundles $E \to M$ (i.e., each fibre E_x has a positive definite inner product, depending continuously on x); it happens that any vector bundle can be provided with a metric. The *Clifford bundle* $\mathrm{Cl}(E) \to M$ is a vector bundle whose fibre at $x \in M$ is the Clifford algebra $\mathrm{Cl}(E_x)$. The space of sections of $\mathrm{Cl}(E)$ becomes an algebra under the fibrewise Clifford multiplication.

Suppose $P \to M$ is a principal fibre bundle with structure group G (acting freely on P on the right), and V is a vector space carrying a linear representation ρ of G. The *associated vector bundle* is $P \times_\rho V \to M$, where $P \times_\rho V$ is the space of orbits of $P \times V$ under the action $(u, v) \cdot g = (ug, \rho(g^{-1})v)$; its transition functions are of the form $\rho(g_{ij})$, where the g_{ij} are local sections of the principal bundle. Conversely, a vector bundle naturally induces a principal bundle associated with it by employing the same transition functions; in which case G is—in principle—either $\mathrm{GL}(k,\mathbb{R})$ or $\mathrm{GL}(k,\mathbb{C})$.

The point is that nothing is lost by constructing metric vector bundles as associated to principal bundles of the orthogonal or unitary groups. That is, if $E \to M$ is a real (or complex) vector bundle of rank k over M, there always exists a principal bundle with structure group $\mathrm{O}(k)$ $[\mathrm{U}(k)]$ such that $E \simeq P\times_\rho\mathbb{R}^k$ $[E \simeq P\times_\sigma\mathbb{C}^k]$, where $\rho: \mathrm{O}(k) \to \mathrm{End}(\mathbb{R}^k)$ $[\sigma: \mathrm{U}(k) \to \mathrm{End}(\mathbb{C}^k)]$ is the standard representation. This is basically due to polar decomposition, applied to the trivializing maps of E: we can substitute the corresponding isometries for these maps, as the topology of $\mathrm{GL}(k,\mathbb{R})/\mathrm{O}(k)$ $[\mathrm{GL}(k,\mathbb{C})/\mathrm{U}(k)]$ is trivial.

A real vector bundle $E \to M$ is *orientable* if an orientation can be continuously defined on the fibres; this amounts to choosing the transition functions g_{ij} in $\mathrm{SO}(k)$ rather than $\mathrm{O}(k)$. In view of (A.6), we may then attempt to lift the transition functions to $h_{ij} \in \mathrm{Spin}(k)$ such that $\varphi(h_{ij}) = g_{ij}$. If this can be done consistently, we obtain a new principal bundle $P' \to M$ with structure group $\mathrm{Spin}(k)$, so that $E = P'\times_{\rho\varphi}\mathbb{R}^k$; and we also get a double covering map $\mu: P' \to P$ satisfying $\mu(uh) = \mu(u)\varphi(h)$. The pair (P',μ) defines a *spin structure* on E.

There are obstructions to orientability and existence of spin structures; these are the Stiefel–Whitney classes $w_1(E) \in H^1(M,\mathbb{Z}_2)$, which vanishes iff E is orientable, and $w_2(E) \in H^2(M,\mathbb{Z}_2)$, which vanishes iff a spin structure exists; for details, see refs. [19, 20, 35]. M is a *spin manifold* if its tangent bundle is orientable and carries a spin structure.

A wider class of vector bundles is related to the group $\mathrm{Spin}^c(k)$. A spin^c structure on $E \to M$ is given by a Spin^c principal bundle P and an isomorphism $E \simeq P \times_{\mathrm{Spin}^c} \mathbb{R}^k$. We have a homomorphism $\phi: \mathrm{Spin}^c(k) \to \mathrm{SO}(k) : (h,\lambda) \mapsto \varphi(h)$. Thus, a spin bundle is naturally spin^c; but a real vector bundle can be spin^c without being spin. This is the case for the underlying bundle of a complex vector bundle, whose structure group is $\mathrm{U}(k)$. To see this, let $\tau: \mathrm{U}(k) \to \mathrm{SO}(2k)$ be the identification taking $g \in \mathrm{U}(k)$ with $g(e_j) = e^{i\theta_j}e_j$ for a suitable basis $\{e_1,\dots,e_k\}$ of $\mathbb{C}^k$ to $\tau(g): e_j \mapsto (\cos\theta_j)e_j + (\sin\theta_j)f_j$ with $f_j = ie_j$. Then $\tau \times \det: \mathrm{U}(k) \to \mathrm{SO}(2k) \times \mathrm{U}(1)$ lifts [2] to $\sigma: \mathrm{U}(k) \to \mathrm{Spin}^c(2k)$ given by

$$\sigma(g) := \prod_{j=1}^{k} e^{i\theta_j/2}(\cos\tfrac{1}{2}\theta_j + (\sin\tfrac{1}{2}\theta_j)e_jf_j),$$

and $\varphi^c \circ \sigma = \tau \times \det$. The complex two-dimensional projective space is the outstanding example of a spin^c manifold which is not spin.

It can be shown that $E \to M$ is a spin^c bundle iff its second Stiefel–Whitney class $w_2(E) \in H^2(M,\mathbb{Z}_2)$ can be lifted to an element of $H^2(M,\mathbb{Z})$. It turns out that for $\dim M = 3$, M is a spin manifold, and if $\dim M = 4$ with M orientable, then M is a spin^c manifold, i.e., $TM \to M$ is a spin^c bundle [20].

A.8. Suppose $E \to M$ is a complex spin^c bundle with $E = P_E \times_{\varphi^c} \mathbb{C}^k$, where $P_E \to M$ is the principal spin^c bundle with the transition functions of E. Suppose also that S is a complex left module for $\mathbb{C}\mathrm{l}_k$, and ρ is the unitary action, by left Clifford multiplication, of $\mathrm{Spin}^c(k)$ on S. Then we can form the *complex spinor bundle*

$$S(E) := P_E \times_\rho S. \tag{A.16}$$

An important complex spinor bundle is the Clifford bundle $\mathbb{C}\mathrm{l}(E)$ itself. If E has rank k, then $\mathrm{Spin}^c(k)$ acts on $\mathbb{C}\mathrm{l}_k$ by $\mathrm{Ad}(g)c = gcg^{-1}$. We can therefore create the spinor bundle $P_E \times_{\mathrm{Ad}} \mathbb{C}\mathrm{l}_k$. Now $\mathrm{Ad}(-1) = \mathrm{id}$, so Ad drops to a representation σ of $\mathrm{SO}(k) \times \mathrm{U}(1)$ and we can replace P_E by the principal $\mathrm{SO}(k) \times \mathrm{U}(1)$ bundle P'_E with the same transition functions as E. Now $\mathrm{Ad}(g)x = \varphi^c(g)x$ for $x \in \mathbb{C}^k$, so that σ is just the usual action of $\mathrm{SO}(k) \times \mathrm{U}(1)$ on $\mathbb{C}^k$. The associated bundle $P'_E \times_\sigma \mathbb{C}\mathrm{l}_k$ is just the Clifford bundle $\mathbb{C}\mathrm{l}(E)$.

Let $\Gamma(S(E))$ be the space of smooth sections of $S(E)$; these are in natural one-to-one correspondence with the functions $s: P_E \to S(E)$, satisfying $s(ug) = \rho(g^{-1})s(u)$, for $u \in P$, $g \in \mathrm{Spin}^c(k)$. Now let $\kappa \in \Gamma(\mathbb{C}\mathrm{l}(E))$; then $\rho(\kappa)s$ makes sense as a function from P_E to $S(E)$, and $(\rho(\kappa)s)(ug) := \rho(\kappa(ug))s(ug) = \rho(g^{-1}\kappa(u)g)\rho(g)^{-1}s(u) = \rho(g^{-1})(\rho(\kappa))s(u)$, so $\rho(\kappa)s \in \Gamma(S(E))$ also. We conclude that $S(E)$ is a bundle of modules over the bundle of algebras $\mathbb{C}\mathrm{l}(E)$, and that $\Gamma(S(E))$ is a module over $\Gamma(\mathbb{C}\mathrm{l}(E))$.

If $n = 2k$ is even, an irreducible complex spinor bundle $S(E)$ carries a natural $\mathbb{Z}_2$-grading. Since E is oriented, each fibre $\mathbb{C}\mathrm{l}(E_x)$ of $\mathbb{C}\mathrm{l}(E)$ contains a canonically determined chirality element γ_x given by (A.13): let γ denote this section of $\mathbb{C}\mathrm{l}(E)$. Then $S(E) = S^+(E) \oplus S^-(E)$ where γ acts by ± 1 on $S^\pm(E)$.

A.9. We recall some facts about connections on vector bundles, mainly to establish notations. The full story can be found, e.g., in ref. [31].

A (linear) connection on a vector bundle $E \to M$ is a linear map $\nabla: \Gamma(E) \to \Gamma(E \otimes T^*M)$ which satisfies

$$\nabla(sf) = (\nabla s)f + s \otimes df \quad \text{for } f \in C^\infty(M). \tag{A.17}$$

The tangent bundle of a Riemannian manifold M carries a distinguished connection—the Levi-Civita connection—which is compatible with the metric and is torsion-free. Consider $\mathrm{Cl}(M) := \mathrm{Cl}(TM)$, the Clifford bundle of M. It may be shown that there is a canonical connection, also called a Levi-Civita connection and denoted by ∇, on $\mathrm{Cl}(M)$ such that

$$\nabla(c_1c_2) = (\nabla c_1)c_2 + c_1(\nabla c_2) \quad \text{for } c_1, c_2 \in \Gamma(\mathrm{Cl}(M)). \tag{A.18}$$

[On the right, $\Gamma(\mathrm{Cl}(M) \otimes T^*M)$ is a bimodule for $\Gamma(\mathrm{Cl}(M))$ in the obvious way.] Moreover, if $S \to M$ is a spinor bundle on M, there is a canonical connection on S, also denoted by ∇, such that $\nabla(c(\kappa)s) = c(\nabla\kappa)s + c(\kappa)\nabla s$ for

$\kappa \in \Gamma(\mathrm{Cl}(M))$, $s \in \Gamma(S)$. These linear connections are related via a principal connection on P_{TM}, obtained by lifting a principal connection on the orthonormal frame bundle of M. Analogous results hold for complex spinor bundles, with $\mathrm{Cl}(M)$ replaced by its complexification $\mathbb{C}\mathrm{l}(M) := \mathbb{C}\mathrm{l}(TM)$.

If S is *the* spinor bundle associated to the complex irreducible module for $\mathbb{C}\mathrm{l}_n$, the canonical connection on S satisfies the three properties:

(1) $\nabla(c(\kappa)s) = c(\nabla\kappa)s + c(\kappa)\nabla s$ for $c \in \Gamma(\mathbb{C}\mathrm{l}(M))$, $s \in \Gamma(S)$.

(2) $(c(X)s \mid s') + (s \mid c(X)s') = 0$ for $X \in \Gamma(TM)$, $s, s' \in \Gamma(S)$. Here $(\cdot \mid \cdot)$ is the hermitian form on S defined on the fibres in section A.4.

(3) $(\nabla_X s \mid s') + (s \mid \nabla_X s') = X(s \mid s')$ for $X \in \Gamma(TM)$, $s, s' \in \Gamma(S)$. Here $\nabla_X : \Gamma(S) \to \Gamma(S)$ is the contraction of ∇ with the vector field X.

Property (2) holds since the Clifford action of TM_x on S_x is skew-adjoint, by the remarks preceding (A.12).

For more general bundles S of modules over $\mathbb{C}\mathrm{l}(M)$, we take (1), (2), (3) as axioms.

A.10. Let a Riemannian metric on M be given; denote by $\xi: TM \to T^*M$ the bundle isomorphism induced by the metric. With it, we identify TM and T^*M.

Definition. Let $S \to M$ be a complex spinor bundle, a left module for $\mathbb{C}\mathrm{l}(M)$, and let $\nabla: \Gamma(S) \to \Gamma(S \otimes TM)$ be the canonical connection on S. There is a morphism of vector bundles $m: S \otimes TM \to S$ given by restriction of the Clifford multiplication to TM. The *Dirac operator* is the mapping $D := m \circ \nabla : \Gamma(S) \to \Gamma(S)$.

Proposition A.7. *Let $\{e_1, \dots, e_n\}$ be an orthonormal frame on an open set $U \subset M$, and let s be a smooth section of S over U. Abbreviating $\nabla_j := \nabla_{e_j}$, we have*

$$Ds = \sum_{j=1}^{n} c(e_j)\nabla_j s. \tag{A.19}$$

Proof. Let $\{t_1, \dots, t_r\}$ be a trivialization of S over U. The connection is given locally by a matrix of one-forms ω_{qp}, such that $\nabla(t_p) = \sum_{q=1}^{n} t_q \otimes \xi(\omega_{qp})$. If now $s = \sum_p t_p f_p \in \Gamma(S)$, we have

$$\nabla(s) = \sum_{p=1}^{n}\Bigl(t_p \otimes \xi(df_p) + \sum_{q=1}^{n} t_q \otimes \xi(\omega_{qp}) f_p\Bigr). \tag{A.20}$$

Thus

$$Ds = \sum_{p=1}^{n}\Bigl(c(\xi(df_p))t_p + \sum_{q=1}^{n} c(\xi(\omega_{qp}))t_q f_p\Bigr)$$

$$= \sum_{j,p=1}^{n} \Big(df_p(e_j)\, c(e_j) t_p + \sum_{q=1}^{n} \omega_{qp}(e_j)\, c(e_j) t_q f_p \Big)$$

$$= \sum_{j=1}^{n} c(e_j) \nabla_j \Big(\sum_p t_p f_p \Big), \tag{A.21}$$

since $\xi(\omega) = \sum_j \omega(e_j)\, e_j$ for any one-form ω. □

As an immediate consequence of (A.19), we note that when $n = 2k$, D is an odd operator on the $\mathbb{Z}_2$-graded spinor bundle, i.e., $D: \Gamma(S^{\pm}) \to \Gamma(S^{\mp})$.

Examples.

(1) If $M = \mathbb{R}^n$, $S = \mathbb{R}^n \times S_n$ where S_n is the irreducible complex module for $\mathbb{C}\mathrm{l}_n$, we have $Ds = \sum_{j=1}^{n} c(e_j)\, \partial_j s$. In particular, if $n = 1$, $\mathbb{C}\mathrm{l}_1 = \mathbb{C}$, $S_1 = \mathbb{C}$, and $D = \mathrm{i}\,\partial/\partial x$. Its kernel $\ker D$ consists of the constant functions on $\mathbb{R}$.

(2) If $M = \mathbb{R}^4$, we have $\mathbb{C}\mathrm{l}_4 = \mathbb{C}^{4\times 4}$, and so $D = \gamma^\mu \partial_\mu$, where $\gamma^\mu = c(e_\mu)$ are the four 4×4 matrices generating the Clifford algebra $\mathbb{C}\mathrm{l}_4$, i.e., satisfying $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2\delta^{\mu\nu}$. Thus $D = \not\partial$ is essentially the operator introduced by Dirac.

(3) The spinor bundle S on which the Dirac operator $D = m \circ \nabla$ acts does not have to be irreducible. An interesting example is when S is the Clifford bundle $\mathbb{C}\mathrm{l}(M) = \mathbb{C}\mathrm{l}(T^*M)$ itself, regarded as a $\mathbb{C}\mathrm{l}(M)$ module under left multiplication. We may alternatively describe this module as the complexified bundle of differential forms $\Lambda^\bullet_{\mathbb{C}} T^*M$ on which $\mathrm{Cl}(M)$ acts by $c(\alpha)\beta = \alpha \wedge \beta - i(\alpha)\beta$ for $\alpha \in \Lambda^1(M)$. Now one can invoke the properties of the Levi-Civita connection ∇ to verify that $e \circ \nabla = d$, the exterior derivative on $\Gamma(\Lambda^\bullet_{\mathbb{C}} T^*M) = \mathcal{E}^\bullet(M)$, and that $i \circ \nabla = -d^*$, the formal adjoint to d.

The Dirac operator on $\mathcal{E}^\bullet(M)$ is thus $D = d + d^*$. Its square is $D^2 = dd^* + d^*d$, the so-called Hodge Laplacian on M. Since d raises and d^* lowers degree, we have $\ker D = \ker D^2$, which is the space of harmonic forms on M. Since M is compact, Hodge's theorem assures us that for each k, $\ker D \cap \mathcal{E}^k(M)$ is isomorphic to the de Rham cohomology group $H^k_{\mathrm{dR}}(M;\mathbb{R})$, which is finite dimensional.

A.11. Denote by $L^2(S)$ the Hilbert space of square-integrable sections of S, the completion of $\Gamma(S)$ with the norm

$$\|s\|^2 := \int_M (s(x) \mid s(x))\, \mu(\mathrm{d}x), \tag{A.22}$$

where $\mu(\mathrm{d}x)$ is the canonical Riemannian volume element on M. We wish to establish:

Theorem A.8. *The Dirac operator is formally self-adjoint on $L^2(S)$.*

The *divergence* $\operatorname{div} X$ of a smooth vector field X on M is the function in $C^\infty(M)$ such that $(\operatorname{div} X)\mu = \mathcal{L}_X(\mu)$, where $\mathcal{L}_X$ is the Lie derivative with respect to X. If $\{e_1, \dots, e_n\}$ is an orthonormal frame on an open subset U of M such that $\nabla e_j = 0$ on U for each j, then we have on U

$$\begin{aligned} \mathcal{L}_X\mu &= d(i(X)\mu) = d\Big(\sum_j (-)^{j-1}(e_j \mid X)\, e_1 \wedge \cdots \wedge \widehat{e}_j \wedge \cdots \wedge e_n\Big) \\ &= \sum_j e_j\,(e_j \mid X)\mu = \sum_j (\nabla_j e_j \mid X)\mu + (e_j \mid \nabla_j X)\mu \end{aligned} \tag{A.23}$$

and from $\nabla e_j = 0$ we conclude that

$$\operatorname{div} X = \sum_{j=1}^{n} (e_j \mid \nabla_j X). \tag{A.24}$$

[The assumption $\nabla e_j = 0$ can always be made; if need be, we can replace the e_j by the vector fields obtained by parallel translation of an orthonormal frame along the geodesics through a given point.]

If $f \in C^\infty(M)$, we therefore have $\operatorname{div}(fX) = Xf + f \operatorname{div} X$. Since M is without boundary, Stokes' theorem gives $\int_M (\operatorname{div} X)\,\mu = \int_M d(i(X)\mu) = 0$. We conclude that

$$\int_M Xf = -\int_M f \operatorname{div} X. \tag{A.25}$$

Proof of theorem A.8 We prove that for $s, s' \in \Gamma(S)$ we have $(Ds \mid s') = (s \mid Ds')$, with $(\cdot \mid \cdot)$ denoting the inner product of $L^2(S)$. We can assume, by using partitions of unity on M, that both s and s' have their support inside an open set $U \subset M$ on which there is an orthonormal frame $\{e_1, \dots, e_n\}$ such that $\nabla e_j = 0$. Reminding the "axioms" for ∇, we arrive at

$$\begin{aligned} (Ds \mid s') &= \sum_j \int_M (c(e_j)\nabla_j s \mid s') = -\sum_j \int_M (\nabla_j s \mid c(e_j)s') \\ &= -\sum_j \int_M e_j(s \mid c(e_j)s') + \sum_j \int_M (s \mid \nabla_j(c(e_j)s')) \\ &= \sum_j \int_M (\operatorname{div} e_j)\,(s \mid c(e_j)s') + \sum_j \int_M (s \mid \nabla_j(c(e_j)s')). \end{aligned} \tag{A.26}$$

From (A.24), $\operatorname{div} e_j = 0$ for the chosen parallel frame. Thus, the formal adjoint $D^\dagger$ of D satisfies

$$D^\dagger s = \sum_{j=1}^{n} \nabla_j(c(e_j)s) = \sum_{j=1}^{n} c(\nabla_j e_j)s + c(e_j)\,\nabla_j s = Ds, \tag{A.27}$$

again since $\nabla e_j = 0$. □

By examining the domain of D more closely, it can moreover be shown that D is essentially self-adjoint; see, for instance, ref. [31, thm. II.5.7]. The Dirac operator may now be redefined to be the closure of D, which is a self-adjoint operator on the Hilbert space $L^2(S)$.

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